A COMPUTATIONAL NOTE ABOUT FRICKE-MACBEATH'S CURVE

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Abstract. The well known Hurwitz upper bound states that a closed Riemann surface S of genus $g \geq 2$ has at most $84(g-1)$ conformal automorphisms. If S has exactly $84(g-1)$ conformal automorphisms, then it is called a Hurwitz curve. The first two genera for which there are Hurwitz's curves are $g \in \{3,7\}$. In both situations there is exactly one such curve up to conformal equivalence, in particular, in both cases the field of moduli is $\mathbb{Q}$. As these two curves are quasiplatonic curves, they are definable over $\mathbb{Q}$. The Hurwitz's curve of genus $g=3$ is given by Klein's quartic $x^3y+y^3z+z^3x=0$. The Hurwitz's curve of genus $g=7$ is known as Fricke-Macbeath's curve and equations over $\mathbb{Q}(\rho)$, where $\rho=e^{2\pi i/7}$, are known due to Macbeath. Unfortunately, explicit equations over $\mathbb{Q}$ are not easy to find for this curve. In this paper we first explain how to construct an explicit model Z_2 of Fricke-Macbeath's curve over $\mathbb{Q}(\sqrt{-7})$ and an explicit isomorphism $L_1 : X \to Z_2$, defined over $\mathbb{Q}(\rho)$. Next, using that explicit model we construct another explicit isomorphism $L_2 : Z_2 \to W$, defined over $\mathbb{Q}(\sqrt{-7})$, where W is some algebraic curve defined over $\mathbb{Q}$. Unfortunately, the equations for W are quite long to write down, but everything is explained in order to perform the computations in a computer.

1. Introduction

Let S be a closed Riemann surface of genus $g \geq 2$. It is well known that $|\mathrm{Aut}(S)| \leq 84(g-1)$ (Hurwitz's upper bound). If happens that $|\mathrm{Aut}(S)| = 84(g-1)$, then one says that S is a Hurwitz's curve. In this case, $S/\mathrm{Aut}(S)$ is an orbifold with signature $(0;2,3,7)$, that is, $S=\mathbb{H}^2/\Gamma$, where Γ is a torsion free normal subgroup of finite index in the triangular Fuchsian group $\Delta = \langle x,y : x^2=y^3=(xy)^7=1\rangle$ acting on the hyperbolic plane $\mathbb{H}^2$ as isometries. As already noticed by Wiman [6], in genera $g=2,4,5,6$ there are no Hurwitz's curves and that for $g=3$ there is exactly one (up to holomorphic equivalence); this being Klein's quartic $x^3y+y^3z+z^3x=0$; whose automorphisms group is the simple group $\mathrm{PSL}(2,7)$ (of order 168).

In genus $g=7$ there is only one (up to conformal equivalence) Hurwitz's curve, called Fricke-Macbeath's curve [4]. It follows from this uniqueness that the field of moduli of Fricke-Macbeath's curve is $\mathbb{Q}$, the field of rational numbers. As quasiplatonic curves can be defined over their fields of modui [8] and Hurwitz's curve are quasiplatonic curves, it follows that Fricke-Macbeath's curve can be defined over $\mathbb{Q}$. It seems that in the literature no equations over $\mathbb{Q}$ are written for this curve. The automorphisms group of Fricke-Macbeath's curve is the simple group $\mathrm{PSL}(2,8)$, consisting of 504 symmetries. In [4] Macbeath computed the following explicit equations over $\mathbb{Q}(\rho)$, where $\rho=e^{2\pi i/7}$, for Fricke-Macbeath's curve involving three

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particular elliptic curves as follows:

$$X = \left\{ \begin{array}{l} y_1^2 = (x-1)(x-\rho^3)(x-\rho^5)(x-\rho^6) \\ \\ y_2^2 = (x-\rho^2)(x-\rho^4)(x-\rho^5)(x-\rho^6) \\ \\ y_4^2 = (x-\rho)(x-\rho^3)(x-\rho^4)(x-\rho^5) \end{array} \right\} \subset \mathbb{C}^4. \tag{1.1}$$

In the talk [7] there is a misprint for the first of the elliptic curves in equations. In the above model is easy to see a group $G \cong \mathbb{Z}_2^3$ of holomorphic automorphisms of X generated by $A_1(x, y_1, y_2, y_4) = (x, -y_1, y_2, y_4)$, $A_2(x, y_1, y_2, y_4) = (x, y_1, -y_2, y_4)$, and $A_3(x, y_1, y_2, y_4) = (x, y_1, y_2, -y_4)$.

In Section 5 we provide a rough explanation about the elliptic curves in the above equations (different from the approach in [4]) in geometric terms of the highest regular branched Abelian cover of the orbifold X/G of signature $(0; 2, 2, 2, 2, 2, 2, 2)$.

An automorphism of order 7 of Fricke-Macbeath's curve is given in such model by

$$B(x, y_1, y_2, y_4) = \left(\rho x, \rho^2 y_2, \rho^2 y_4, \rho^2 \frac{y_1 y_2}{(x-\rho^5)(x-\rho^6)} \right).$$

The automorphism B normalizes G and it induces, on the orbifold $X/G = \widehat{\mathbb{C}}$, the rotation $T(x) = \rho x$; moreover, $X/\langle G, B\rangle$ has signature $(0; 2, 7, 7)$, that is, the group $\langle G, B\rangle$ defines a regular dessin d'enfants (X, β), where $\beta(x, y_1, y_2, y_4) = x^7$ (called an Edmods map) [3, 7]. Such a dessin d'enfants is seen to be defined over $\mathbb{Q}(\rho)$, but it is known to be definable over $\mathbb{Q}(\sqrt{-7})$ [3]. As a direct consequence of our computations, we obtain an isomorphic dessin defined over $\mathbb{Q}(\sqrt{-7})$ (see Remark 3.1 in Section 3).

Notice that X also admits the following anticonformal involution

$$J(x, y_1, y_2, y_4) = \left(\frac{1}{\overline{x}}, \frac{\overline{y_1}}{\overline{x}^2}, \frac{\rho^5 \overline{y_2}}{\overline{x}^2}, \frac{\rho^3 \overline{y_4}}{\overline{x}^2} \right).$$

We may see that $JBJ = B$ and $JA_jJ = A_j$, for $j = 1, 2, 4$. We also have the regular dessin d'enfants (X, δ), where $\delta(x, y_1, y_2, y_4) = 1/x^7$. As $\delta = C \circ \beta \circ J$, where $C(x) = \overline{x}$, we have that these two dessins are chirals (these are the two dessins in genus 7 whose graph is the complete graph K_8 appearing in [3]).

In this paper we first describe a theoretical/computational method which permits to obtain an explicit a birational isomorphism, defined over $\mathbb{Q}(\rho)$)

$$L : X \to W$$

so that W satisfies that $W^\sigma = W$ for every $\sigma \in \mathrm{Gal}(\mathbb{Q}(\rho)/\mathbb{Q})$; that is, W is defined over $\mathbb{Q}$. This method is based on the constructive proof of Weil's Galois descent theorem [2]. In order to get explicitly L, one needs to find an explicit set of generators of the invariant polynomials under a suitable linear cyclic group of order 6 acting by permutations on a 24-dimensional space. Unfortunately, it is hard to get (inclusive with MAGMA [10]) such a set of generators. Instead to proceed in a direct way, we divide this search into two parts. In the first one we explain how to provide (with the help of a computer) an explicit algebraic curve Z_2 defined over $\mathbb{Q}(\sqrt{-7})$ together an explicit isomorphism $L_1^* : X \to Z_2$ (also we provide explicitly its inverse $(L_1^*)^{-1} : Z_2 \to X$). This is the first known model of Fricke-Macbeath's curve over $\mathbb{Q}(\sqrt{-7})$ to our knowledge. Also, the isomorphism L_1^*

provides an isomorphism between the dessin (X, β) with the dessin (Z_2, β^*), where β^* is defined over $\mathbb{Q}$, that is, the dessin is defined over $\mathbb{Q}(\sqrt{-7})$. Next, using the explicit model Z_2, we provide an explicit isomorphism $L_2 : Z_2 \to W$, where W is defined over $\mathbb{Q}$ (we are able to see that W is defined over $\mathbb{Q}$ without the knowledge of equations for it). As both, Z_2 and L_2 are explicitly given, equations for W over $\mathbb{Q}$ are possible to compute (with the help of a computer), but unfortunately they are long to write them in this paper. Now, $L = L_2 \circ L_1^* : X \to W$ is an explicit isomorphism as desired.

Recently, Bradley Brock told me that he was able to compute a plane equation for Fricke-Macbeath's equation over $\mathbb{Q}$ as $1+7xy+21x^2y^2+35x^3y^3+28x^4y^4+2x^7+2y^7=0$. To obtain the model he used the canonical model in Macbeath's original paper [4] and then got a suitable change of variables. Moreover, he asserted that the jacobian is isogenous to E^7 where $j(E) = 1792$ and Edoes not have Complex Multiplication.

2. In search of a birational isomorphism

In this section we explain the general ideas in the search of an isomorphism $L : X \to W$, where W is defined over $\mathbb{Q}$. The same ideas will be used in the next sections to obtain it explicitly by working it in two steps.

The Galois extension $\mathbb{Q}(\rho)/\mathbb{Q}$ has as Galois group $\Gamma = \mathrm{Gal}(\mathbb{Q}(\rho)/\mathbb{Q}) = \langle\sigma\rangle \cong \mathbb{Z}_6$, where $\sigma(\rho) = \rho^3$. In particular, $\sigma^2(\rho) = \rho^2$, $\sigma^3(\rho) = \rho^6$, $\sigma^4(\rho) = \rho^4$ and $\sigma^5(\rho) = \rho^5$.

2.1. **A weil's datum for X.** All the curves X, X^σ, X^{σ^2}, X^{σ^3}, X^{σ^4} and X^{σ^5} are birationally equivalent. In fact, we may describe explicit birational isomorphisms as follows.

$$f_\sigma : X \to X^\sigma$$
$$(x, y_1, y_2, y_4) \mapsto \left(\frac{1}{x}, \frac{y_1}{x^2}, \frac{\rho y_2 y_4}{x^2(x-\rho^4)(x-\rho^5)}, \frac{\rho^2 y_2}{x^2}\right)$$

$$f_{\sigma^2} : X \to X^{\sigma^2}$$
$$(x, y_1, y_2, y_4) \mapsto \left(x, y_1, y_4, \frac{y_2 y_4}{(x-\rho^4)(x-\rho^5)}\right)$$

$$f_{\sigma^3} : X \to X^{\sigma^3}$$
$$(x, y_1, y_2, y_4) \mapsto \left(\frac{1}{x}, \frac{y_1}{x^2}, \frac{\rho^2 y_2}{x^2}, \frac{\rho^4 y_4}{x^2}\right)$$

$$f_{\sigma^4} : X \to X^{\sigma^4}$$
$$(x, y_1, y_2, y_4) \mapsto \left(x, y_1, \frac{y_2 y_4}{(x-\rho^4)(x-\rho^5)}, y_2\right)$$

$$f_{\sigma^5} : X \to X^{\sigma^5}$$
$$(x, y_1, y_2, y_4) \mapsto \left(\frac{1}{x}, \frac{y_1}{x^2}, \frac{\rho^4 y_4}{x^2}, \frac{\rho y_2 y_4}{x^2(x-\rho^4)(x-\rho^5)}\right)$$

It can be easily checked that the following Weil's co-cycle condition holds

$$f_{\sigma_1\sigma_2} = f_{\sigma_2}^{\sigma_1} \circ f_{\sigma_1}, \text{ for every } \sigma_1, \sigma_2 \in \Gamma.$$

In fact, the last co-cycle condition was used to define f_{σ^j} starting from f_σ. In particular, $f_e = I$, where $e = \sigma^0$ is the identity of $\mathrm{Gal}(\mathbb{Q}(\rho)/\mathbb{Q})$ and I is the identity automorphism of X. In other words, the collection of birational isomorphisms $\{f_{\sigma^j}\}_{j=0}^{5}$ is a Weil's datum for X (this is another way to see that X is definable over $\mathbb{Q}$ as a consequence of Weil's Galois descent theorem [5]).

2.2. **Another model for X.** Let us consider the rational map

$$\begin{gathered}\Phi : X \subset \mathbb{C}^4 \to \mathbb{C}^{24}\\ (x, y_1, y_2, y_4) \mapsto (\vec{x}, \vec{z}, \vec{w}, \vec{u}, \vec{v}, \vec{r})\end{gathered}$$

where

$$\vec{x} = (x_1, x_2, x_3, x_4) = (x, y_1, y_2, y_4)$$

$$\vec{z} = (z_1, z_2, z_3, z_4) = f_\sigma(\vec{x}) = \left(\frac{1}{x}, \frac{y_1}{x^2}, \frac{\rho y_2 y_4}{x^2(x-\rho^4)(x-\rho^5)}, \frac{\rho^2 y_2}{x^2}\right)$$

$$\vec{w} = (w_1, w_2, w_3, w_4) = f_{\sigma^2}(\vec{x}) = \left(x, y_1, y_4, \frac{y_2 y_4}{(x-\rho^4)(x-\rho^5)}\right)$$

$$\vec{u} = (u_1, u_2, u_3, u_4) = f_{\sigma^3}(\vec{x}) = \left(\frac{1}{x}, \frac{y_1}{x^2}, \frac{\rho^2 y_2}{x^2}, \frac{\rho^4 y_4}{x^2}\right)$$

$$\vec{v} = (v_1, v_2, v_3, v_4) = f_{\sigma^4}(\vec{x}) = \left(x, y_1, \frac{y_2 y_4}{(x-\rho^4)(x-\rho^5)}, y_2\right)$$

$$\vec{r} = (r_1, r_2, r_3, r_4) = f_{\sigma^5}(\vec{x}) = \left(\frac{1}{x}, \frac{y_1}{x^2}, \frac{\rho^4 y_4}{x^2}, \frac{\rho y_2 y_4}{x^2(x-\rho^4)(x-\rho^5)}\right)$$

It turns out that $\Phi : X \to \Phi(X)$ is a birational isomorphism (the inverse is just given by the projection on the $\vec{x}$-coordinate).

Equations defining the algebraic curve $\Phi(X)$ are the following ones

$$\Phi(X) = \left\{\begin{array}{l}
x_2^2 = (x_1-1)(x_1-\rho^3)(x_1-\rho^5)(x_1-\rho^6)\\[2mm]
x_3^2 = (x_1-\rho^2)(x_1-\rho^4)(x_1-\rho^5)(x_1-\rho^6)\\[2mm]
x_4^2 = (x_1-\rho)(x_1-\rho^3)(x_1-\rho^4)(x_1-\rho^5)\\[2mm]
z_1 = \dfrac{1}{x_1},\ z_2 = \dfrac{x_2}{x_1^2},\ z_3 = \dfrac{\rho x_3 x_4}{x_1^2(x_1-\rho^4)(x_1-\rho^5)},\ z_4 = \dfrac{\rho^2 x_3}{x_1^2},\\[2mm]
w_1 = x_1,\ w_2 = x_2,\ w_3 = x_4,\ w_4 = \dfrac{x_3 x_4}{(x_1-\rho^4)(x_1-\rho^5)},\\[2mm]
u_1 = \dfrac{1}{x_1},\ u_2 = \dfrac{x_2}{x_1^2},\ u_3 = \dfrac{\rho^2 x_3}{x_1^2},\ u_4 = \dfrac{\rho^4 x_4}{x_1^2},\\[2mm]
v_1 = x_1,\ v_2 = x_2,\ v_3 = \dfrac{x_3 x_4}{(x_1-\rho^4)(x_1-\rho^5)},\ v_4 = x_3,\\[2mm]
r_1 = \dfrac{1}{x_1},\ r_2 = \dfrac{x_2}{x_1^2},\ r_3 = \dfrac{\rho^4 x_4}{x_1^2},\ r_4 = \dfrac{\rho x_3 x_4}{x_1^2(x_1-\rho^4)(x_1-\rho^5)}
\end{array}\right\} \tag{2.1}$$

2.3. **A permutation action.** Each $\tau \in \Gamma$ induces a natural bijection

$$\widehat{\tau} : \mathbb{C}^n \to \mathbb{C}^n : (y_1, \ldots, y_n) \mapsto (\sigma(y_1), \ldots, \sigma(y_n)).$$

Let us consider the natural permutation action of Γ on the coordinates of $\mathbb{C}^{24}$ defined by

$$\Theta(\sigma)(\vec{x}, \vec{z}, \vec{w}, \vec{u}, \vec{v}, \vec{r}) = (\vec{z}, \vec{w}, \vec{u}, \vec{v}, \vec{r}, \vec{x})$$

Let us notice that the stabilizer G of $\Phi(X)$ is trivial since

$$G = \{\tau \in \Gamma : \Theta(\tau)(\Phi(X)) = \Phi(X)\} = \{\tau \in \Gamma : X^{\tau} = X\} = \{e\}.$$

Next, we should observe that, for each $\tau \in \Gamma$, the following diagram commutes [2]

$$\begin{array}{ccl} X & \overset{\Phi}{\rightarrow} & \Phi(X) \\ \downarrow_{f_\tau} & & \downarrow_{\Theta(\tau)} \\ X^{\tau} & \overset{\Phi^{\tau}}{\rightarrow} & \Theta(\tau)(\Phi(X)) = \Phi^{\tau}(X^{\tau}) = \Phi(X)^{\tau} \\ \downarrow_{\widehat{\tau}^{-1}} & & \downarrow_{\widehat{\tau}^{-1}} \\ X & \overset{\Phi}{\rightarrow} & \Phi(X) \end{array} \tag{2.2}$$

Similarly, it is not difficult to see that, for every $\eta, \tau \in \Gamma$, we have that

$$(*) \quad \Theta(\eta) \circ \widehat{\tau} = \widehat{\tau} \circ \Theta(\eta).$$

2.4. **A birational isomorphism.** Assume we have computed a set of generators of the algebra of polynomials invariants under the linear action $\Theta(\Gamma)$, say t_1,..., t_N. It is not difficult to note that we may choose each $t_j \in \mathbb{Q}[\vec{x}, \vec{z}, \vec{w}, \vec{u}, \vec{v}, \vec{r}]$. Next, we construct the rational map

$$\Psi : \mathbb{C}^{24} \to \mathbb{C}^{N}$$
$$(\vec{x}, \vec{z}, \vec{w}, \vec{u}, \vec{v}, \vec{r}) \mapsto (t_1, \ldots, t_N)$$

It can be checked, for each $\tau \in \Gamma$, the following equalities:

$$\begin{array}{l} \Psi^{\tau} = \Psi \\ \Psi \circ \Theta(\tau) = \Psi \end{array} \tag{2.3}$$

Also, it holds (as we have chosen a set of generators of the invariant polynomials for the action of $\Theta(\Gamma)$) that Ψ is a branched regular cover with Galois group Γ.

It turns out that, if we set $W = \Psi(\Phi(X))$ and $L = \Psi \circ \Psi$, then

$$L : X \to W$$

is a birational isomorphism (since the stabilizer G of $\Phi(X)$ is trivial).

2.5. W **can be defined by polynomials over** $\mathbb{Q}$**.** If $\tau \in \Gamma$, then

$$W^{\tau} = L(X)^{\tau} = L^{\tau}(X^{\tau}) = \Psi^{\tau} \circ \Phi^{\tau}(X^{\tau}) = \Psi \circ \Theta(\tau)(\Phi(X)) = \Psi \circ \Phi(X) = L(X) = W.$$

This last set of equalities asserts that W can be defined by polynomials with coefficient over $\mathbb{Q}$. In fact, it is almost clear that (using L and the equations for X) that W is defined by a set of polynomial equations over $\mathbb{Q}(\rho)$; say by $P_1, \ldots, P_r \in \mathbb{Q}(\rho)[t_1, \ldots, t_{24}]$. The above set of equalities is telling us that, for each $\tau \in \Gamma$, the

polynomials $P_1^\tau,\ldots,P_r^\tau \in \mathbb{Q}(\rho)[t_1,\ldots,t_{24}]$ also are null over W. Let us consider the basis $\{1,\rho,\rho^2,\rho^3,\rho^4,\rho^5\}$ of the $\mathbb{Q}$-vector space $\mathbb{Q}(\rho)$. The polynomials

$$\mathrm{Tr}(P_j), \mathrm{Tr}(\rho P_j), \mathrm{Tr}(\rho^2 P_j), \mathrm{Tr}(\rho^3 P_j), \mathrm{Tr}(\rho^4 P_j), \mathrm{Tr}(\rho^5 P_j) \in \mathbb{Q}[t_1,\ldots,t_{24}]$$

also define W as their common zero locus, where

$$\mathrm{Tr}(aQ) = \sum_{l=0}^{5} \sigma^l(a) Q^{\sigma^l}$$

2.6. **In search of a set of invariant polynomials.** The program MAGMA [10] may be used to obtain an explicit generating set of invariant polynomials. Unfortunately, in our case this was not possible in my computer (Mac OSX 10.5.8 2X2.66 GHz Dual-Core Intel Xeon 3GB 667 MHz DDR2 FB-DIMM).

By hand, we may find a collection of invariant polynomials given by

$$\begin{aligned} t_j &= x_j + z_j + w_j + u_j + v_j + r_j; j = 1,2,3,4 \\ t_{4+j} &= x_j^2 + z_j^2 + w_j^2 + u_j^2 + v_j^2 + r_j^2; j = 1,2,3,4 \\ t_{8+j} &= x_j^3 + z_j^3 + w_j^3 + u_j^3 + v_j^3 + r_j^3; j = 1,2,3,4 \end{aligned}$$

but I am not sure if they form a complete set of generators.

So, the idea will be to work in two steps. The first one will consist in produce a model for Fricke-Macbeath's curve in the degree two extension $\mathbb{Q}(\sqrt{-7})$ and then to use that model to obtain a model over $\mathbb{Q}$. In the next two sections we describe these two steps.

3. Step 1: A model of Fricke-Macbeath's curve over $\mathbb{Q}(\sqrt{-7})$

In this section we will explain how to obtain an explicit model for Fricke-Macbeath's curve over the quadratic extension $\mathbb{Q}(\sqrt{-7})$. Mostly of the computations have been carry out with MAGMA [10] and with MATHEMATICA [9].

Let $\tau = \sigma^2$ (an element of order 3) and $\mathbb{Z}_3 \cong N = \langle \tau \rangle \lhd \Gamma$. The fixed field of N is $\mathbb{Q}(\rho+\rho^2+\rho^4) = \mathbb{Q}(\sqrt{-7})$.

Let us consider the rational map

$$\Phi_1 : X \to \mathbb{C}^{12}$$

$$(x, y_1, y_2, y_4) \mapsto (\vec{x}, \vec{w}, \vec{v})$$

Again, we may see that Φ_1 produces a birational isomorphism between X and $\Phi_1(X)$ (its inverse is just the projection on the $\vec{x}$-coordinate; just as for Φ).

Equations defining the algebraic curve $\Phi_1(X)$ are the following ones

$$\Phi_1(X) = \left\{ \begin{array}{l} x_2^2 = (x_1-1)(x_1-\rho^3)(x_1-\rho^5)(x_1-\rho^6) \\ \\ x_3^2 = (x_1-\rho^2)(x_1-\rho^4)(x_1-\rho^5)(x_1-\rho^6) \\ \\ x_4^2 = (x_1-\rho)(x_1-\rho^3)(x_1-\rho^4)(x_1-\rho^5) \\ \\ w_1 = x_1,\ w_2 = x_2,\ w_3 = x_4,\ w_4 = \dfrac{x_3 x_4}{(x_1-\rho^4)(x_1-\rho^5)}, \\ \\ v_1 = x_1,\ v_2 = x_2,\ v_3 = \dfrac{x_3 x_4}{(x_1-\rho^4)(x_1-\rho^5)},\ v_4 = x_3 \end{array} \right\} \tag{3.1}$$

Next, we consider the linear permutation action of N on the coordinates of $\mathbb{C}^{12}$ defined by

$$\Theta_1(\tau)(\vec{x},\vec{w},\vec{v}) = (\vec{w},\vec{v},\vec{x})$$

Again, the stabilizer of $\Phi_1(X)$ is just the trivial group.

A generating set of invariant polynomials for the previous linear action can be obtained with MAGMA as

$$\begin{array}{rcl}
t_1 & = & x_1 + w_1 + v_1 \\
t_2 & = & x_2 + w_2 + v_2 \\
t_3 & = & x_3 + w_3 + v_3 \\
t_4 & = & x_4 + w_4 + v_4 \\
t_5 & = & x_1^2 + w_1^2 + v_1^2 \\
t_6 & = & x_2^2 + w_2^2 + v_2^2 \\
t_7 & = & x_3^2 + w_3^2 + v_3^2 \\
t_8 & = & x_4^2 + w_4^2 + v_4^2 \\
t_9 & = & x_1^3 + w_1^3 + v_1^3 \\
t_{10} & = & x_2^3 + w_2^3 + v_2^3 \\
t_{11} & = & x_3^3 + w_3^3 + v_3^3 \\
t_{12} & = & x_4^3 + w_4^3 + v_4^3
\end{array}$$

The map

$$\Psi_1 : \mathbb{C}^{12} \to \mathbb{C}^{12}$$

$$(\vec{x},\vec{w},\vec{v}) \mapsto (t_1, ..., t_{12})$$

satisfies the following properties:

$$\begin{array}{l}
\Psi_1^{\tau^j} = \Psi_1,\ j = 0,1,2; \\
\Psi_1 \circ \Theta_1(\tau^j) = \Psi_1,\ j = 0,1,2.
\end{array} \tag{3.2}$$

Also, it holds (as we have chosen a set of generators of the invariant polynomials for the action of $\Theta_1(N)$) that Ψ_1 is a branched regular cover with Galois group N. As in the previous case, it turns out that, if we set $Z_1 = \Psi_1(\Phi_1(X))$ and $L_1 = \Psi_1 \circ \Psi_1$, then

$$L_1 : X \to Z_1$$

is a birational isomorphism (since the stabilizer of $\Phi_1(X)$ is trivial).

Similarly, if $\eta \in N$, then

$$Z_1^\eta = L_1(X)^\eta = L_1^\eta(X^\eta) = \Psi_1^\eta \circ \Phi_1^\eta(X^\eta) = \Psi_1 \circ \Theta_1(\eta)(\Phi_1(X)) = \Psi_1 \circ \Phi_1(X) = L_1(X) = Z,$$

so Z_1 can be defined by polynomials with coefficient over $\mathbb{Q}(\sqrt{-7})$.

We have tried to compute directly equations for Z_1 using MAGMA and the explicit form of L_1, but unfortunately we couldn't get an answer (it is very heavy computational task). We will proceed in the search of such equation by hands.

It is clear that

$$x_1 = \frac{t_1}{3}$$

$$x_2 = \frac{t_2}{3}$$

$$t_4 = t_3$$

$$(*)\ x_4 = \frac{(t_3 - x_3)(\frac{t_1}{3} - \rho^4)(\frac{t_1}{3} - \rho^5)}{x_3 + (\frac{t_1}{3} - \rho^4)(\frac{t_1}{3} - \rho^5)}$$

$$t_5 = \frac{t_1^2}{3}$$
$$t_6 = \frac{t_2^2}{3}$$
$$t_8 = t_7$$
$$(**)\ x_4^2 = \frac{(t_7 - x_3^2)(\frac{t_1}{3} - \rho^4)^2(\frac{t_1}{3} - \rho^5)^2}{x_3^2 + (\frac{t_1}{3} - \rho^4)^2(\frac{t_1}{3} - \rho^5)^2}$$
$$t_9 = \frac{t_1^3}{9}$$
$$t_{10} = \frac{t_2^3}{9}$$
$$t_{12} = t_{11}$$
$$(***)\ x_4^3 = \frac{(t_{11} - x_3^3)(\frac{t_1}{3} - \rho^4)^3(\frac{t_1}{3} - \rho^5)^3}{x_3^3 + (\frac{t_1}{3} - \rho^4)^3(\frac{t_1}{3} - \rho^5)^3}$$

Equality $(*)$ permits to obtain x_4 uniquely in terms of t_1 and x_3.

Equation
$$x_2^2 = (x_1 - 1)(x_1 - \rho^3)(x_1 - \rho^5)(x_1 - \rho^6)$$
asserts the polynomial equation (relating t_1 and t_2)
$$P_1(t_1, t_2, t_3, t_7, t_{11}) = -81 + 27(1 + (\rho + \rho^2 + \rho^4))t_1 + 9t_1^2 - 3(\rho + \rho^2 + \rho^4)t_1^3 - t_1^4 + 9t_2^2 = 0.$$

Notice that $P_1(t_1, t_2, t_3, t_7, t_{11}) \in \mathbb{Q}(\sqrt{-7})[t_1, t_2, t_3, t_7, t_{11}]$.

From the equation
$$x_3^2 = (x_1 - \rho^2)(x_1 - \rho^4)(x_1 - \rho^5)(x_1 - \rho^6)$$
we obtain the equation
$$(1)\ x_3^2 = (t_1 - 3\rho^2)(t_1 - 3\rho^4)(t_1 - 3\rho^5)(t_1 - 3\rho^6)/81$$

From the equation
$$x_4^2 = (x_1 - \rho)(x_1 - \rho^3)(x_1 - \rho^4)(x_1 - \rho^5)$$
we obtain the equation
$$(2)\ x_4^2 = (t_1 - 3\rho)(t_1 - 3\rho^3)(t_1 - 3\rho^4)(t_1 - 3\rho^5)/81$$

In this way, by replacing the above values for x_3^2 and x_4^2 (obtained in (1) and (2)) in the above equality $(**)$, we obtain the polynomial equation
$$P_2(t_1, t_2, t_3, t_7, t_{11}) = 27 + 27(\rho + \rho^2 + \rho^4) - 18t_1 - 3(1 + (\rho + \rho^2 + \rho^4))t_1^2 - 2t_1^3 - t_1^4 + 27t_7 = 0.$$

Notice that $P_2(t_1, t_2, t_3, t_7, t_{11}) \in \mathbb{Q}(\sqrt{-7})[t_1, t_2, t_3, t_7, t_{11}]$.

Also, if we replace, in equality $(***)$ the values of x_3^3 by $x_3(x_1 - \rho^2)(x_1 - \rho^4)(x_1 - \rho^5)(x_1 - \rho^6)/81$ and x_4^3 by $x_4(t_1 - 3\rho)(t_1 - 3\rho^3)(t_1 - 3\rho^4)(t_1 - 3\rho^5)/81$, where x_4 is given in $(*)$, then we obtain a polynomial which is of degree one in the variable x_3.

$x_3 = (-9\rho^2(-162t_1 - 18t_1^3 + 4t_1^5 - 243(1 + t_{11}) + t_1^2(27 - 54t_3) + 6t_1^4t_3) + 3(729 + 18t_1^4 - 6t_1^5 - 27t_1^3(-6 + t_3) - t_1^6(-2 + t_3) + 243t_1(3 + t_3) + 81t_1^2(2 + t_{11} + t_3)) + \rho^3(2187 - t_1^7 + 27t_1^4(-6 + t_3) + 9t_1^5(-3 + t_3) + 486t_1^2t_3 + 81t_1^3(1 + t_3) + 729t_1(1 + 2t_3)) + \rho^5(2187 + 27t_1^4 + 12t_1^6 + t_1^7 - 729t_1(-1 + t_{11} - t_3) + 729t_1^2t_3 + 81t_1^3(5 + t_3) + 9t_1^5(1 + 2t_3)) + \rho(2916t_1 + 3t_1^6 - t_1^7 - 81t_1^3(-6 + t_3) - 2187(-2 + t_3) - 27t_1^4(-2 + t_3) + 9t_1^5(2 + t_3) + 243t_1^2(5 + 2t_3)) + \rho^4(2187 + t_1^7 - 729t_1(-3 + t_{11} - 2t_3) - 81t_1^3(-1 + t_3) + 27t_1^4(1 + t_3) + 9t_1^5(-1 + 2t_3) + 243t_1^2(1 + 3t_3)))/(9(t_1^5 - 243t_{11} + 27t_1^2(-1 + t_3) + 81t_1t_3 + 9t_1^3t_3 +$

$3t_1^4t_3 + \rho(3+t_1)(-81+18t_1^2-9t_1^3+2t_1^4+27t_1t_3)+27\rho^2t_1(3+t_1^2+t_1(3+t_3))+\rho^4t_1(243+3t_1^3+t_1^4+9t_1^2(-1+t_3)+27t_1(3+t_3))+\rho^5(-6t_1^4+t_1^5+243(1+t_3)+81t_1(2+t_3)+9t_1^3(2+t_3)+27t_1^2(3+t_3))+\rho^3t_1(162+36t_1^2+6t_1^3+2t_1^4+27t_1(4+t_3))))$

Then, using $(*)$, we obtain

$x_4 = -((3\rho^4-t_1)(3\rho^5-t_1)(-\rho^3(-2187-729t_1+t_1^7+243t_1^2t_3(2+t_3)+9t_1^5(3+t_3)+27t_1^4(6+t_3)+81t_1^3(-1+3t_3))+\rho^4(2187+27t_1^4+t_1^7+9t_1^5(-1+t_3)-729t_1(-3+t_{11}+t_3)-243t_1^2(-1+t_3^2)-81t_1^3(-1+t_3^2))+\rho(4374+486t_1^3+54t_1^4+3t_1^6-t_1^7-9t_1^5(-2+t_3)-243t_1^2(-5+t_3^2)-729t_1(-4-t_3+t_3^2))-3(t_1^6(-2+t_3)+3t_1^5(2+t_3)-729(1+t_{11}t_3)-81t_1^2(2+t_{11}+2t_3-t_3^2)+9t_1^4(-2+t_3^2)+243t_1(-3-t_3+t_3^2)+27t_1^3(-6+t_3+t_3^2))-9\rho^2(4t_1^5-243(1+t_{11})+81t_1(-2+t_3)+6t_1^4t_3+9t_1^3(-2+3t_3)+27t_1^2(1+t_3+t_3^2))+\rho^5(12t_1^6+t_1^7-243t_1^2t_3^2+9t_1^5(1+t_3)+27t_1^4(1+2t_3)-81t_1^3(-5+t_3+t_3^2)-2187(-1+t_3+t_3^2)-729t_1(-1+t_{11}+t_3+t_3^2))))/(9(567t_1^3+6t_1^6+t_1^7+\rho(-3+t_1)(-54t_1^3+t_1^6+9t_1^4(-4+t_3)+729(-2+t_3)+243t_1(-2+t_3)-81t_1^2(-2+t_3))+27t_1^4(-7+t_3)+9t_1^5(-5+t_3)+2187(2+t_3)+243t_1^2(-1+2t_3)+729t_1(1+2t_3)+\rho^5(2187+216t_1^4+3t_1^6+2t_1^7+729t_1t_3+729t_1^2(1+t_3)+18t_1^5(2+t_3)+81t_1^3(16+t_3))+\rho^3t_1(9t_1^5+t_1^6+27t_1^3(-4+t_3)+9t_1^4(3+t_3)+81t_1^2(5+t_3)+729(-5+2t_3)+243t_1(-3+2t_3))+\rho^4(2187+6t_1^6+2t_1^7-81t_1^3(-14+t_3)+18t_1^5(-2+t_3)+1458t_1t_3+27t_1^4(5+t_3)+243t_1^2(1+3t_3))-9\rho^2(-243+243t_1-27t_1^3+t_1^5-54t_1^2(-5+t_3)+t_1^4(-9+6t_3)))).$

Now, using such values for x_3 and x_4, and replacing them in (1) and (2) above, we obtain another two polynomials identities $P_1(t_1,t_3,t_7,t_{11})=0$ and $P_2(t_1,t_3,t_7,t_{11})=0$, where the polynomials are defined over $\mathbb{Q}(\rho)$ (see the Appendix). In this way,

$$Z_1 = \left\{ \begin{array}{c} t_4=t_3,\ 3t_5=t_1^2,\ 3t_6=t_2^2,\ t_8=t_9 \\ 9t_9=t_1^3,\ 9t_{10}=t_2^3,\ t_{12}=t_{11} \\ P_1(t_1,t_2,t_3,t_7,t_{11})=0 \\ P_2(t_1,t_2,t_3,t_7,t_{11})=0 \\ P_3(t_1,t_2,t_3,t_7,t_{11})=0 \\ P_4(t_1,t_2,t_3,t_7,t_{11})=0 \end{array} \right\} \subset \mathbb{C}^{12}$$

Notice that, by the above computations, we have explicitly the inverse of L_1 given as

$$L_1^{-1}: Z_1 \to X$$
$$(t_1,...,t_{12}) \mapsto (x_1,x_2,x_3,x_4),$$

that is, we may write x_1, x_2, x_3 and x_4 in terms of t_1,..., t_{12} (in fact, only in terms of t_1, t_2, t_3, t_7 and t_{11}).

As the variables t_1,..., t_{12} are uniquely determined only by the variables t_1, t_2, t_3, t_7 and t_{11}, if we consider the projection

$$\pi: \mathbb{C}^{12} \to \mathbb{C}^5$$
$$(t_1,...,t_{12}) \mapsto (t_1,t_2,t_3,t_7,t_{11}),$$

then

$$L_1^* = \pi \circ L_1 : X \to Z_2$$
$$L_1^*(x,y_1,y_2,y_4)$$
$$||$$
$$\left(3x, 3y_1, y_2+y_4+\frac{y_2y_4}{(x-\rho^4)(x-\rho^5)}, y_2^2+y_4^2+\frac{y_2^2y_4^2}{(x-\rho^4)^2(x-\rho^5)^2}, y_2^3+y_4^3+\frac{y_2^3y_4^3}{(x-\rho^4)^3(x-\rho^5)^3}\right)$$

is a birational isomorphism, where

$$Z_2 = \left\{ \begin{array}{c} P_1(t_1,t_2,t_3,t_7,t_{11}) = 0 \\ P_2(t_1,t_2,t_3,t_7,t_{11}) = 0 \\ P_3(t_1,t_2,t_3,t_7,t_{11}) = 0 \\ P_4(t_1,t_2,t_3,t_7,t_{11}) = 0 \end{array} \right\} \subset \mathbb{C}^5$$

The inverse $(L_1^*)^{-1} : Z_2 \to X$ is given as

$$(L_1^*)^{-1}(t_1,t_2,t_3,t_7,t_{11}) = (x_1,x_2,x_3,x_4),$$

where x_1,..., x_4 are given by the previous formulae.

As $Z_1^\eta = Z_1$, for every $\eta \in N$, as as π is defined over $\mathbb{Q}$, we see that $Z_2^\eta = Z_2$, for every $\eta \in N$, that is, Z_2 can be defined by polynomials over $\mathbb{Q}(\sqrt{-7})$. To obtain such equations over $\mathbb{Q}(\sqrt{-7})$, we replace each polynomial P_j ($j = 3,4$) by the polynomials (with coefficients in $\mathbb{Q}(\sqrt{-7})$)

$$Q_{j,1} = \mathrm{Tr}(P_j),\ Q_{j,2} = \mathrm{Tr}(\rho P_j),\ Q_{j,3} = \mathrm{Tr}(\rho^2 P_j)$$

that is

$$Z_2 = \left\{ \begin{array}{c} P_1(t_1,t_2,t_3,t_7,t_{11}) = 0 \\ P_2(t_1,t_2,t_3,t_7,t_{11}) = 0 \\ P_3(t_1,t_2,t_3,t_7,t_{11}) + P_3(t_1,t_2,t_3,t_7,t_{11})^\tau + P_3(t_1,t_2,t_3,t_7,t_{11})^{\tau^2} = 0 \\ \rho P_3(t_1,t_2,t_3,t_7,t_{11}) + \rho^2 P_3(t_1,t_2,t_3,t_7,t_{11})^\tau + \rho^4 P_3(t_1,t_2,t_3,t_7,t_{11})^{\tau^2} = 0 \\ \rho^2 P_3(t_1,t_2,t_3,t_7,t_{11}) + \rho^4 P_3(t_1,t_2,t_3,t_7,t_{11})^\tau + \rho P_3(t_1,t_2,t_3,t_7,t_{11})^{\tau^2} = 0 \\ P_4(t_1,t_2,t_3,t_7,t_{11}) + P_4(t_1,t_2,t_3,t_7,t_{11})^\tau + P_4(t_1,t_2,t_3,t_7,t_{11})^{\tau^2} = 0 \\ \rho P_4(t_1,t_2,t_3,t_7,t_{11}) + \rho^2 P_4(t_1,t_2,t_3,t_7,t_{11})^\tau + \rho^4 P_4(t_1,t_2,t_3,t_7,t_{11})^{\tau^2} = 0 \\ \rho^2 P_4(t_1,t_2,t_3,t_7,t_{11}) + \rho^4 P_4(t_1,t_2,t_3,t_7,t_{11})^\tau + \rho P_4(t_1,t_2,t_3,t_7,t_{11})^{\tau^2} = 0 \end{array} \right\} \subset \mathbb{C}^5$$

We have obtained an explicit model Z_2 for Fricke-Macbeath's curve over $\mathbb{Q}(\sqrt{-7})$ together explicit isomorphisms $L_1^* : X \to Z_2$ and $(L_1^*)^{-1} : Z_2 \to X$.

Remark 3.1. *Notice that the regular dessin d'enfants (X,β), given in the Introduction, is isomorphic to that provided by the pair (Z_2,β^*), where $\beta^*(t_1,t_2,t_3,t_7,t_{11}) = (t_1/3)^7$; that is, such a dessin d'enfants is defined over $\mathbb{Q}(\sqrt{-7})$.*

4. Step 2: A model of Fricke-Macbeath's curve over $\mathbb{Q}$

Let us now consider the explicit model $Z_2 \subset \mathbb{C}^5$ over $\mathbb{Q}(\sqrt{-7})$, constructed in the previous section. We keep the same notations. Let $M = \mathrm{Gal}(\mathbb{Q}(\sqrt{-7})/\mathbb{Q}) = \langle \eta \rangle \cong \mathbb{Z}_2$, where η is the complex conjugation.

As already noticed in the Introduction, X admits the following anticonformal involution

$$J(x,y_1,y_2,y_4) = \left(\frac{1}{\overline{x}}, \frac{\overline{y_1}}{\overline{x}^2}, \frac{\rho^5 \overline{y_2}}{\overline{x}^2}, \frac{\rho^3 \overline{y_4}}{\overline{x}^2} \right)$$

In this way, $T = L_1^* \circ J \circ (L_1^*)^{-1}$ is an anticonformal involution of Z_2. It is not difficult to see that by setting $g_e = I$ and $g_\eta = S \circ T$, where $S(t_1,t_2,t_3,t_7,t_{11}) = (\overline{t_1},\overline{t_2},\overline{t_3},\overline{t_7},\overline{t_{11}})$, we obtain a Weil's datum for Z_2.

Now, identically as done before, we consider the rational map

$$\Phi_2 : Z_2 \to \mathbb{C}^{10}$$

$$(t_1, t_2, t_3, t_7, t_{11}) \mapsto (t_1, t_2, t_3, t_7, t_{11}, s_1, s_2, s_3, s_7, s_{11})$$

where $g_\eta(t_1, t_2, t_3, t_7, t_{11}) = (s_1, s_2, s_3, s_7, s_{11})$. Then, again we may see that Φ_2 induces a birational isomorphism between Z_2 and $\Phi_2(Z_2)$.

In this case,

$$\Phi_2(Z_2) = \left\{ \begin{array}{c} Q_{1,1}(t_1, t_2, t_3, t_7, t_{11}) = \cdots = Q_{4,3}(t_1, t_2, t_3, t_7, t_{11}) = 0 \\ g_\eta(t_1, t_2, t_3, t_7, t_{11}) = (s_1, s_2, s_3, s_7, s_{11}) \end{array} \right\} \subset \mathbb{C}^{10}.$$

In this case, the Galois group M induces the permutation action $\Theta_2(M)$ defined as

$$\Theta(\eta)(t_1, t_2, t_3, t_7, t_{11}, s_1, s_2, s_3, s_7, s_{11}) = (s_1, s_2, s_3, s_7, s_{11}, t_1, t_2, t_3, t_7, t_{11})$$

A set of generators for the invariant polynomials (with respect to the previous permutation action) is given by

$$q_1 = t_1 + s_1,\ q_2 = t_2 + s_2,\ q_3 = t_3 + s_3,$$
$$q_4 = t_7 + s_7,\ q_5 = t_{11} + s_{11},\ q_6 = t_1^2 + s_1^2,$$
$$q_7 = t_2^2 + s_2^2,\ q_8 = t_3^2 + s_3^2,\ q_9 = t_7^2 + s_7^2,$$
$$q_{10} = t_{11}^2 + s_{11}^2$$

Then the rational map

$$\Psi_2 : \mathbb{C}^{10} \to \mathbb{C}^{10}$$

$$(t_1, t_2, t_3, t_7, t_{11}, s_1, s_2, s_3, s_7, s_{11}) \mapsto (q_1, q_2, q_3, q_4, q_5, q_6, q_7 q_8, q_9, q_{10})$$

satisfies the following properties:

$$\begin{array}{l} \Psi_2^\eta = \Psi_2; \\ \Psi_2 \circ \Theta_2(\eta) = \Psi_2. \end{array} \tag{4.1}$$

There are two possibilities:

(1) $\Phi_2(Z_2) = \Phi_2(Z_2)$; in which case $Z_2^\eta = Z_2$ and Z_2 will be already defined over $\mathbb{Q}$ (which seems not to be the case); and
(2) the stabilizer of $\Phi_2(Z_2)$ under $\Theta_2(M)$ is trivial.

Under the assumption (2) above, we have that $\Psi_2 : \Phi_2(Z_2) \to W = \Psi_2(\Phi_2(Z_2))$ is a birational isomorphism and that, as before, W is defined over $\mathbb{Q}$. That is, the map $L_2 = \Psi_2 \circ \Phi_2 : Z_2 \to W$ is an explicit birational isomorphism and W is defined over $\mathbb{Q}$. In this way, $L = L_2 \circ L_1^* : X \to W$ is an explicit birational isomorphism as desired at the beginning.

Now, again as in the previous section, as R_2 and Z_2 are explicitly given, one may compute explicit equations for W over $\mathbb{Q}(\sqrt{-7})$, say by the polynomials $A_1, ..., A_m \in \mathbb{Q}(\sqrt{-7})[q_1, ..., q_{10}]$ (this may be done with MAGMA [10] or by hands, but computations are heavy and very long). We do not write these equations as they are large expressions to be written down in this paper (anyway, all of this can be done explicitly with the computer). To obtain equations over $\mathbb{Q}$ we replace each A_j (which is not already defined over $\mathbb{Q}$) by the traces $A_j + A_j^\eta$, $iA_j - iA_j^\eta$.

5. A remark on the elliptic curves in (1.1)

5.1. **A connection to homology covers.** Let us set $\lambda_1 = 1$, $\lambda_2 = \rho$, $\lambda_3 = \rho^2$, $\lambda_4 = \rho^3$, $\lambda_5 = \rho^4$, $\lambda_6 = \rho^5$ and $\lambda_7 = \rho^6$. It is known that if S is Fricke-Macbeath's curve, then it admits a regular branched cover $Q : S \to \widehat{\mathbb{C}}$ whose branch locus is the set $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7\}$ and deck group $G \cong \mathbb{Z}_2^3$. Let us consider a Fuchsian group

$$\Gamma = \langle \alpha_1, ..., \alpha_7 : \alpha_1^2 = \cdots = \alpha_7^2 = \alpha_1\alpha_2 \cdots \alpha_7 = 1 \rangle$$

acting on the hyperbolic plane $\mathbb{H}^2$ uniformizing the orbifold S/G.

If Γ' denotes the derived subgroup of Γ, then Γ' acts freely and $\widehat{S} = \mathbb{H}^2/\Gamma'$ is a closed Riemann surface. Let $H = \Gamma/\Gamma' \cong \mathbb{Z}_2^6$; a group of conformal automorphisms of $\widehat{S}$. Then there exists a set of generators of H, say a_1,..., a_6, so that the only elements of H acting with fixed points are these and $a_7 = a_1a_2a_3a_4a_5a_6$. In fact, $\widehat{S}$ corresponds to the algebraic curve (called the homology cover of S/H)

$$\widehat{S} = \left\{ \begin{array}{rcl} x_1^2 + x_2^2 + x_3^2 & = & 0 \\ \left(\dfrac{\lambda_3 - 1}{\lambda_4 - 1}\right) x_1^2 + x_2^2 + x_4^2 & = & 0 \\ \\ \left(\dfrac{\lambda_4 - 1}{\lambda_5 - 1}\right) x_1^2 + x_2^2 + x_5^2 & = & 0 \\ \\ \left(\dfrac{\lambda_5 - 1}{\lambda_6 - 1}\right) x_1^2 + x_2^2 + x_6^2 & = & 0 \\ \\ \left(\dfrac{\lambda_6 - 1}{\lambda_7 - 1}\right) x_1^2 + x_2^2 + x_7^2 & = & 0 \end{array} \right\} \subset \mathbb{P}^6_{\mathbb{C}},$$

and a_j is just multiplication by -1 at the coordinate x_j. The regular branched cover $P : \widehat{S} \to \widehat{\mathbb{C}}$ is given bt

$$P([x_1 : x_2 : x_3 : x_4 : x_5 : x_6 :: x_7]) = \frac{x_2^2 + x_1^2}{x_2^2 + \lambda_7 x_1^2} = z.$$

The branch locus of P is given by the set of the $7th$-roots of unity $\{\lambda_1, ..., \lambda_7\}$. All details can be found in [1].

By classical covering theory, there should be a subgroup $K < H$, $K \cong \mathbb{Z}_2^3$, acting freely on $\widehat{S}$ so that there is an isomorphism $\phi : S \to \widehat{S}/K$ with $\phi G \phi^{-1} = H/K$.

Let us also observe that the rotation $R(z) = \rho z$ lifts under P to an automorphism T of $\widehat{S}$ of order 7 of the form

$$T([x_1 : \cdots : x_7]) = [c_1x_7 : c_2x_1 : c_3x_2 : c_4x_3 : c_5x_4 : c_6x_5 : c_7x_6]$$

for suitable comples numbers c_j. One has that $Ta_jT^{-1} = a_{j+1}$, for $j = 1, ..., 6$ and $Ta_7T^{-1} = a_1$. The subgroup K above must satisfies that $TKT^{-1} = K$ as R lifts to Fricke-Macbeath's curve (as noticed in the Introduction).

5.2. **About the elliptic curves in Fricke-Macbeath's curve.** The subgroup $K^* = \langle a_1a_3a_7, a_2a_3a_5, a_1a_2a_4 \rangle \cong \mathbb{Z}_2^3$ acts freely on $\widehat{S}$ and it is normalized by T. In particular, $S^* = \widehat{S}/K$ is a closed Riemann surface of genus 7 admitting the group $L = H/K^* = \{e, a_1^*, ..., a_7^*\} \cong \mathbb{Z}_2^3$ (where a_j^* is the involution induced by a_j) as a group of automorphisms and it also has an automorphism T^* of order 7

(induced by T) permuting cyclically the involutions a_j^*. As $S^*/\langle L, T^*\rangle = \widehat{S}/\langle H, T\rangle$ has signature $(0; 2, 7, 7)$, we may see that $S = S^*$ and $K = K^*$.

We may see that $L = \langle a_1^*, a_2^*, a_3^*\rangle$ and $a_4^* = a_1^* a_2^*$, $a_5^* = a_2^* a_3^*$, $a_6^* = a_1^* a_2^* a_3^*$ and $a_7^* = a_1^* a_3^*$. In this way, we may see that every involution of H/K is induced by one of the involutions (and only one) with fixed points; so every involutions in L acts with 4 fixed points.

Let $a_i^*, a_j^* \in H/K$ be any two different involutions, so $\langle a_i^*, a_j^*\rangle \cong \mathbb{Z}_2^2$. Then, by Riemann-Hurwitz, the quotient surface $S/\langle a_i^*, a_j^*\rangle$ is a closed Riemann surface of genus 1 with six cone points of order 2. These six cone points are projected onto three of the cone points of S/H. These points are λ_i, λ_j and λ_r, where $a_r^* = a_i^* a_j^*$. In this way, the genus one surface is given by the elliptic curve

$$y^2 = \prod_{k \notin \{i,j,r\}} (x - \lambda_k)$$

So, for instance, if we consider $i = 2$ and $j = 3$, then $r = 5$ and the elliptic curve is

$$y_1^2 = (x-1)(x-\rho^3)(x-\rho^5)(x-\rho^6).$$

If $i = 1$ and $j = 2$, then $r = 4$ and the elliptic curve is

$$y_2^2 = (x-\rho^2)(x-\rho^4)(x-\rho^5)(x-\rho^6).$$

If $i = 1$ and $j = 3$, then $r = 7$ and the elliptic curve is

$$y_4^2 = (x-\rho)(x-\rho^3)(x-\rho^4)(x-\rho^5).$$

We have obtained the three elliptic curves appearing in the Fricke-Macbeath's curve (1.1).

5.3. **Another model for Fricke-Macbeath's curve.** The above description of Fricke-Macbeath's curve in terms of the homology cover $\widehat{S}$ permits to obtain an explicit model. Let us consider now an affine model of $\widehat{S}$, say by taking $x_7 = 1$, with we denote by $\widehat{S}^0$. In this case the involution a_7 is multiplication of all coordinates by -1. A set of generators for the algebra of invariant polynomials in $\mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6]$ under the natural linear action induced by K is

$$t_1 = x_1^2, t_2 = x_2^2, t_3 = x_3^2, t_4 = x_4^2, t_5 = x_5^2, t_6 = x_6^2, t_7 = x_1x_2x_5, t_8 = x_1x_2, x_3x_6$$

$$t_9 = x_1x_4x_6, t_{10} = x_1x_3x_4x_5, t_{11} = x_2x_4x_5x_6, t_{12} = x_2x_3x_4, t_{13} = x_3x_5x_6.$$

If we set

$$F : \widehat{S}^0 \to \mathbb{C}^{13}$$

$$F(x_1, x_2, x_3, x_4, x_5, x_6) = (t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}),$$

then $F(\widehat{S}^0)$ will provide a model for Fricke-Macbeath's curve S (affine). Equations for such an affine model of S are

$$\left\{\begin{array}{c}
t_1+t_2+t_3=0\\
\left(\frac{\lambda_3-1}{\lambda_4-1}\right)t_1+t_2+t_4=0\\
\left(\frac{\lambda_4-1}{\lambda_5-1}\right)t_1+t_2+t_5=0\\
\left(\frac{\lambda_5-1}{\lambda_6-1}\right)t_1+t_2+t_6=0\\
\left(\frac{\lambda_6-1}{\lambda_7-1}\right)t_1+t_2+1=0\\
t_6t_{10}=t_9t_{13},\; t_6t_7t_{12}=t_8t_{11},\; t_5t_9t_{12}=t_{10}t_{11}\\
t_5t_8=t_7t_{13},\; t_5t_6t_{12}=t_{11}t_{13},\; t_4t_8=t_9t_{12}\\
t_4t_7t_{13}=t_{10}t_{11},\; t_4t_6t_7=t_9t_{11},\; t_3t_{11}=t_{12}t_{13}\\
t_3t_6t_7=t_8t_{13},\; t_3t_5t_9=t_{10}t_{13},\; t_3t_5t_6=t_{13}^2\\
t_3t_4t_7=t_{10}t_{12},\; t_2t_{10}=t_7t_{12},\; t_2t_9t_{13}=t_8t_{11}\\
t_2t_5t_9=t_7t_{11},\; t_2t_4t_{13}=t_{11}t_{12},\; t_2t_4t_5t_6=t_{11}^2\\
t_2t_3t_9=t_8t_{12},\; t_2t_3t_4=t_{12}^2,\; t_1t_{12}t_{13}=t_8t_{10}\\
t_1t_{11}=t_7t_9,\; t_1t_6t_{12}=t_8t_9,\; t_1t_5t_{12}=t_7t_{10}\\
t_1t_4t_{13}=t_9t_{10},\; t_1t_4t_6=t_9^2,\; t_1t_3t_4t_5=t_{10}^2\\
t_1t_2t_{13}=t_7t_8,\; t_1t_2t_5=t_7^2,\; t_1t_2t_3t_6=t_8^2
\end{array}\right\}\subset\mathbb{C}^{13}$$

Of course, one may see that the variables t_2, t_3, t_4, t_5 and t_6 are uniquely determined with the variable t_1. Other variables can also be determined in order to get a lower dimensional model.

6. Appendix

$P_3(t_1,t_3,t_7,t_{11}) = 4782969-4782969\rho-19131876\rho^2-4782969\rho^3-4782969\rho^4-4782969\rho^5-1594323t_1-23914845\rho t_1-28697814\rho^2t_1-1594323\rho^5t_1-3188646t_1^2-19663317\rho t_1^2-6908733\rho^2t_1^2+18068994\rho^3t_1^2+15943230\rho^4t_1^2+3188646\rho^5t_1^2-885735t_1^3-885735\rho t_1^3+6731586\rho^2t_1^3+20017611\rho^3t_1^3+15943230\rho^4t_1^3+3424842t_1^4+5609655\rho t_1^4+7144929\rho^2t_1^4+13581270\rho^3t_1^4+7322076\rho^4t_1^4+944784\rho^5t_1^4+3601989t_1^5+3109914\rho t_1^5+4507407\rho^2t_1^5+5235678\rho^3t_1^5+3149280\rho^4t_1^5-177147\rho^5t_1^5+1003833t_1^6+1436859\rho t_1^6+997272\rho^2t_1^6+1082565\rho^3t_1^6+747954\rho^4t_1^6-997272\rho^5t_1^6+492075t_1^7+658287\rho t_1^7+205578\rho^2t_1^7+365229\rho^3t_1^7+96228\rho^4t_1^7-56862\rho^5t_1^7+174231t_1^8+143613\rho t_1^8+104976\rho^2t_1^8+15309\rho^3t_1^8+45927\rho^4t_1^8+5832\rho^5t_1^8+3159t_1^9+24300\rho t_1^9-3645\rho^2t_1^9-25272\rho^3t_1^9-10692\rho^4t_1^9-26973\rho^5t_1^9+2106t_1^{10}+3807\rho t_1^{10}-1944\rho^2t_1^{10}-2349\rho^3t_1^{10}-6723\rho^4t_1^{10}-1701\rho^5t_1^{10}+702t_1^{11}-675\rho t_1^{11}-432\rho^2t_1^{11}-1107\rho^3t_1^{11}-1269\rho^4t_1^{11}-1377\rho^5t_1^{11}+117t_1^{12}-99\rho t_1^{12}-297\rho^2t_1^{12}-189\rho^3t_1^{12}-117\rho^4t_1^{12}-360\rho^5t_1^{12}+24t_1^{13}+15\rho t_1^{13}-36\rho^2t_1^{13}-33\rho^3t_1^{13}+15\rho^4t_1^{13}-27\rho^5t_1^{13}-2t_1^{14}+\rho t_1^{14}-6\rho^2t_1^{14}-5\rho^3t_1^{14}-\rho^4t_1^{14}-\rho^5t_1^{14}-9565938\rho^3t_{11}+9565938\rho^4t_{11}-3188646t_1t_{11}+6377292\rho t_1t_{11}+3188646\rho^2t_1t_{11}+9565938\rho^4t_1t_{11}+6377292\rho^5t_1t_{11}-1062882t_1^2t_{11}+3188646\rho t_1^2t_{11}+5314410\rho^2t_1^2t_{11}+1062882\rho^3t_1^2t_{11}+3188646\rho^4t_1^2t_{11}+3188646\rho^5t_1^2t_{11}-708588t_1^3t_{11}+708588\rho t_1^3t_{11}+1771470\rho^2t_1^3t_{11}+354294\rho^4t_1^3t_{11}+354294\rho^5t_1^3t_{11}-236196t_1^4t_{11}+236196\rho t_1^4t_{11}+590490\rho^2t_1^4t_{11}+118098\rho^4t_1^4t_{11}+118098\rho^5t_1^4t_{11}-78732t_1^5t_{11}+78732\rho t_1^5t_{11}+196830\rho^2t_1^5t_{11}+39366\rho^4t_1^5t_{11}+39366\rho^5t_1^5t_{11}-26244t_1^6t_{11}+26244\rho t_1^6t_{11}+65610\rho^2t_1^6t_{11}+13122\rho^4t_1^6t_{11}+13122\rho^5t_1^6t_{11}-8748t_1^7t_{11}+8748\rho t_1^7t_{11}+21870\rho^2t_1^7t_{11}+4374\rho^3t_1^7t_{11}+4374\rho^5t_1^7t_{11}-1458t_1^8t_{11}+5832\rho^2t_1^8t_{11}-2916\rho^4t_1^8t_{11}-1458\rho^5t_1^8t_{11}-486t_1^9t_{11}-486\rho t_1^9t_{11}-486\rho^3t_1^9t_{11}-972\rho^4t_1^9t_{11}-972\rho^5t_1^9t_{11}+4782969\rho^3t_{11}^2-4782969\rho^4t_{11}^2+1594323t_1t_{11}^2-3188646\rho t_1t_{11}^2-1594323\rho^2t_1t_{11}^2-1594323\rho^3t_1t_{11}^2-3188646\rho^4t_1t_{11}^2-3188646\rho^5t_1t_{11}^2-531441\rho t_1^2t_{11}^2-2125764\rho^2t_1^2t_{11}^2-531441\rho^3t_1^2t_{11}^2-531441\rho^5t_1^2t_{11}^2+177147t_1^3t_{11}^2+177147\rho t_1^3t_{11}^2+$

$177147\rho^3t_1^3t_{11}^2+354294\rho^4t_1^3t_{11}^2+354294\rho^5t_1^3t_{11}^2-19131876t_3-9565938\rho t_3-9565938\rho^2t_3-9565938\rho^3t_3-9565938\rho^4t_3-9565938\rho^5t_3-12754584t_1t_3-6377292\rho t_1t_3+6377292\rho^3t_1t_3-9565938\rho^4t_1t_3-3188646t_1^2t_3-4251528\rho t_1^2t_3+5314410\rho^2t_1^2t_3+4251528\rho^3t_1^2t_3-5314410\rho^4t_1^2t_3-4251528\rho^5t_1^2t_3-1771470t_1^3t_3-2125764\rho t_1^3t_3+1062882\rho^3t_1^3t_3-4251528\rho^4t_1^3t_3-5314410\rho^5t_1^3t_3+236196t_1^4t_3-708588\rho t_1^4t_3-354294\rho^2t_1^4t_3+708588\rho^3t_1^4t_3-1889568\rho^4t_1^4t_3-2125764\rho^5t_1^4t_3+236196t_1^5t_3-314928\rho t_1^5t_3-314928\rho^2t_1^5t_3+39366\rho^3t_1^5t_3-669222\rho^4t_1^5t_3-905418\rho^5t_1^5t_3+78732t_1^6t_3-104976\rho t_1^6t_3-183708\rho^2t_1^6t_3-13122\rho^3t_1^6t_3-209952\rho^4t_1^6t_3-301806\rho^5t_1^6t_3+39366t_1^7t_3-26244\rho t_1^7t_3-56862\rho^2t_1^7t_3+4374\rho^3t_1^7t_3-56862\rho^4t_1^7t_3-87480\rho^5t_1^7t_3+16038t_1^8t_3-7290\rho t_1^8t_3-20412\rho^2t_1^8t_3-2916\rho^3t_1^8t_3-16038\rho^4t_1^8t_3-30618\rho^5t_1^8t_3+4860t_1^9t_3-1458\rho t_1^9t_3-9234\rho^2t_1^9t_3-1944\rho^3t_1^9t_3-4374\rho^4t_1^9t_3-8262\rho^5t_1^9t_3+1944t_1^{10}t_3-162\rho t_1^{10}t_3-2268\rho^2t_1^{10}t_3-486\rho^3t_1^{10}t_3-324\rho^4t_1^{10}t_3-972\rho^5t_1^{10}t_3+270t_1^{11}t_3-54\rho t_1^{11}t_3-594\rho^2t_1^{11}t_3-324\rho^3t_1^{11}t_3+108\rho^4t_1^{11}t_3-162\rho^5t_1^{11}t_3+18t_1^{12}t_3+18\rho t_1^{12}t_3-108\rho^2t_1^{12}t_3-18\rho^3t_1^{12}t_3+54\rho^4t_1^{12}t_3+36\rho^5t_1^{12}t_3+6t_1^{13}t_3+6\rho t_1^{13}t_3+6\rho^3t_1^{13}t_3+12\rho^4t_1^{13}t_3+12\rho^5t_1^{13}t_3-9565938\rho t_{11}t_3+9565938\rho^3t_{11}t_3+3188646t_1t_{11}t_3+3188646\rho t_1t_{11}t_3+6377292\rho^2t_1t_{11}t_3+6377292\rho^3t_1t_{11}t_3+6377292\rho^4t_1t_{11}t_3-3188646\rho^5t_1t_{11}t_3+6377292\rho t_1^2t_{11}t_3+5314410\rho^2t_1^2t_{11}t_3+1062882\rho^3t_1^2t_{11}t_3+1062882\rho^4t_1^2t_{11}t_3+1062882\rho^5t_1^2t_{11}t_3-1771470t_1^3t_{11}t_3-708588\rho t_1^3t_{11}t_3+708588\rho^2t_1^3t_{11}t_3-1417176\rho^3t_1^3t_{11}t_3-2480058\rho^4t_1^3t_{11}t_3-1771470\rho^5t_1^3t_{11}t_3-236196\rho t_1^4t_{11}t_3+118098\rho^3t_1^4t_{11}t_3-118098\rho^4t_1^4t_{11}t_3-590490\rho^5t_1^4t_{11}t_3+39366t_1^5t_{11}t_3+236196\rho t_1^5t_{11}t_3+236196\rho^2t_1^5t_{11}t_3+78732\rho^3t_1^5t_{11}t_3+196830\rho^4t_1^5t_{11}t_3+39366\rho^5t_1^5t_{11}t_3-26244t_1^6t_{11}t_3+13122\rho t_1^6t_{11}t_3+78732\rho^2t_1^6t_{11}t_3-13122\rho^3t_1^6t_{11}t_3-39366\rho^4t_1^6t_{11}t_3-13122\rho^5t_1^6t_{11}t_3-8748t_1^7t_{11}t_3-8748\rho t_1^7t_{11}t_3-8748\rho^3t_1^7t_{11}t_3-17496\rho^4t_1^7t_{11}t_3-17496\rho^5t_1^7t_{11}t_3-4782969t_3^2-4782969\rho t_3^2-9565938\rho^2t_3^2-4782969\rho^3t_3^2-4782969\rho^4t_3^2-4782969\rho^5t_3^2-7971615t_1t_3^2-1594323\rho t_1t_3^2-4782969\rho^2t_1t_3^2-6377292\rho^3t_1t_3^2-1594323\rho^4t_1t_3^2-8503056t_1^2t_3^2-4251528\rho t_1^2t_3^2-1594323\rho^2t_1^2t_3^2-2657205\rho^3t_1^2t_3^2-2657205\rho^4t_1^2t_3^2+1062882\rho^5t_1^2t_3^2-2302911t_1^3t_3^2-2480058\rho t_1^3t_3^2-1417176\rho^2t_1^3t_3^2+177147\rho^3t_1^3t_3^2-354294\rho^4t_1^3t_3^2-1062882\rho^5t_1^3t_3^2+236196t_1^4t_3^2+649539\rho t_1^4t_3^2+236196\rho^3t_1^4t_3^2+1299078\rho^4t_1^4t_3^2+885735\rho^5t_1^4t_3^2-413343t_1^5t_3^2-137781\rho t_1^5t_3^2-275562\rho^3t_1^5t_3^2+413343\rho^5t_1^5t_3^2-91854t_1^6t_3^2-229635\rho t_1^6t_3^2-91854\rho^2t_1^6t_3^2-45927\rho^4t_1^6t_3^2+32805t_1^7t_3^2+17496\rho t_1^7t_3^2-10935\rho^2t_1^7t_3^2+32805\rho^3t_1^7t_3^2+63423\rho^4t_1^7t_3^2+48114\rho^5t_1^7t_3^2-1458t_1^8t_3^2+5832\rho t_1^8t_3^2+2187\rho^2t_1^8t_3^2-2187\rho^3t_1^8t_3^2+5832\rho^4t_1^8t_3^2+20412\rho^5t_1^8t_3^2-1215t_1^9t_3^2-3159\rho t_1^9t_3^2-2673\rho^2t_1^9t_3^2-486\rho^3t_1^9t_3^2-2187\rho^4t_1^9t_3^2+1215\rho^5t_1^9t_3^2+486t_1^{10}t_3^2-486\rho^2t_1^{10}t_3^2+486\rho^3t_1^{10}t_3^2+729\rho^4t_1^{10}t_3^2+486\rho^5t_1^{10}t_3^2+81t_1^{11}t_3^2+81\rho t_1^{11}t_3^2+81\rho^3t_1^{11}t_3^2+162\rho^4t_1^{11}t_3^2+162\rho^5t_1^{11}t_3^2=0$

$P_4(t_1,t_3,t_7,t_{11})=1549681956+1937102445\rho-387420489\rho^2+1937102445\rho^4+387420489\rho^5+1549681956t_1+2066242608\rho t_1-1678822119\rho^2t_1-516560652\rho^3t_1+2711943423\rho^4t_1+2195382771\rho^5t_1+473513931t_1^2+344373768\rho t_1^2-645700815\rho^2t_1^2+688747536\rho^3t_1^2+4347718821\rho^4t_1^2+2927177028\rho^5t_1^2+301327047t_1^3+2424965283\rho t_1^3+1922753538\rho^2t_1^3+3156759540\rho^3t_1^3+5337793404\rho^4t_1^3+2826734679\rho^5t_1^3+975725676t_1^4+3271550796\rho t_1^4+3348078300\rho^2t_1^4+2774122020\rho^3t_1^4+3137627664\rho^4t_1^4+2228863554\rho^5t_1^4+435250179t_1^5+1551276279\rho t_1^5+1729840455\rho^2t_1^5+691936182\rho^3t_1^5+680775921\rho^4t_1^5+379448874\rho^5t_1^5+171124002t_1^6+628694703\rho t_1^6+438970266\rho^2t_1^6+48361131\rho^3t_1^6+11160261\rho^4t_1^6+81841914\rho^5t_1^6+17183259t_1^7+150752097\rho t_1^7+173958354\rho^2t_1^7-86802030\rho^3t_1^7-61292862\rho^4t_1^7+105048171\rho^5t_1^7-65071998t_1^8-41157153\rho t_1^8+16828965\rho^2t_1^8-56568942\rho^3t_1^8-83495286\rho^4t_1^8-30941676\rho^5t_1^8-2775303t_1^9-11475189\rho t_1^9-3680721\rho^2t_1^9+5708070\rho^3t_1^9-12931731\rho^4t_1^9-7361442\rho^5t_1^9+$

$2617839t_1^{10}+3155841\rho t_1^{10}+5911461\rho^2t_1^{10}+3083670\rho^3t_1^{10}+5589972\rho^4t_1^{10}+5865534\rho^5t_1^{10}-2591595t_1^{11}-756702\rho t_1^{11}+1791153\rho^2t_1^{11}+341172\rho^3t_1^{11}-894483\rho^4t_1^{11}+150903\rho^5t_1^{11}-127575t_1^{12}-423549\rho t_1^{12}+316386\rho^2t_1^{12}+898128\rho^3t_1^{12}-311283\rho^4t_1^{12}-352107\rho^5t_1^{12}+$

$219429t_1^{13} + 141183\rho t_1^{13} + 180306\rho^2 t_1^{13} + 275562\rho^3 t_1^{13} + 221130\rho^4 t_1^{13} - 1701\rho^5 t_1^{13} +$
$15633t_1^{14} + 61641\rho t_1^{14} + 43578\rho^2 t_1^{14} + 31752\rho^3 t_1^{14} + 31590\rho^4 t_1^{14} + 11421\rho^5 t_1^{14} +$
$1647t_1^{15} + 5157\rho t_1^{15} + 7209\rho^2 t_1^{15} + 4806\rho^3 t_1^{15} - 3402\rho^4 t_1^{15} + 459\rho^5 t_1^{15} + 729t_1^{16} +$
$324\rho t_1^{16} + 495\rho^2 t_1^{16} + 333\rho^3 t_1^{16} - 288\rho^4 t_1^{16} - 774\rho^5 t_1^{16} + 63t_1^{17} + 108\rho t_1^{17} - 18\rho^2 t_1^{17} - 12\rho^3 t_1^{17}$
$+3\rho^4 t_1^{17} - 39\rho^5 t_1^{17} + t_1^{18} + 5\rho t_1^{18} - \rho^3 t_1^{18} - 2\rho^4 t_1^{18} + 4\rho^5 t_1^{18} - 774840978t_{11} + 774840978\rho^5 t_{11}$
$- 1033121304t_1 t_{11} - 516560652\rho t_1 t_{11} + 1291401630\rho^2 t_1 t_{11} + 774840978\rho^3 t_1 t_{11} +$
$516560652\rho^4 t_1 t_{11} + 774840978\rho^5 t_1 t_{11} - 774840978t_1^2 t_{11} + 258280326\rho t_1^2 t_{11} +$
$1463588514\rho^2 t_1^2 t_{11} + 1635775398\rho^3 t_1^2 t_{11} + 172186884\rho^4 t_1^2 t_{11} - 344373768\rho^5 t_1^2 t_{11} +$
$631351908t_1^3 t_{11} + 688747536\rho t_1^3 t_{11} + 1348797258\rho^2 t_1^3 t_{11} + 1234006002\rho^3 t_1^3 t_{11}$
$- 114791256\rho^4 t_1^3 t_{11} - 373071582\rho^5 t_1^3 t_{11} + 373071582t_1^4 t_{11} + 325241892\rho t_1^4 t_{11} +$
$296544078\rho^2 t_1^4 t_{11} + 47829690\rho^3 t_1^4 t_{11} - 315675954\rho^4 t_1^4 t_{11} - 660049722\rho^5 t_1^4 t_{11} +$
$207261990t_1^5 t_{11} + 149866362\rho t_1^5 t_{11} - 117979902\rho^2 t_1^5 t_{11} - 108413964\rho^3 t_1^5 t_{11}$
$- 255091680\rho^4 t_1^5 t_{11} - 232771158\rho^5 t_1^5 t_{11} + 89282088t_1^6 t_{11} - 1062882\rho t_1^6 t_{11}$
$- 89282088\rho^2 t_1^6 t_{11} - 127545840\rho^3 t_1^6 t_{11} - 80779032\rho^4 t_1^6 t_{11} - 51018336\rho^5 t_1^6 t_{11}$
$- 7085880t_1^7 t_{11} - 19840464\rho t_1^7 t_{11} - 62001450\rho^2 t_1^7 t_{11} - 62001450\rho^3 t_1^7 t_{11}$
$-24800580\rho^4 t_1^7 t_{11} - 15234642\rho^5 t_1^7 t_{11} - 2834352t_1^8 t_{11} - 7203978\rho t_1^8 t_{11} - 14644152\rho^2 t_1^8 t_{11}$
$-10274526\rho^3 t_1^8 t_{11} - 1889568\rho^4 t_1^8 t_{11} + 7085880\rho^5 t_1^8 t_{11} - 1850202t_1^9 t_{11} - 2598156\rho t_1^9 t_{11}$
$-1771470\rho^2 t_1^9 t_{11} - 1850202\rho^3 t_1^9 t_{11} + 1299078\rho^4 t_1^9 t_{11} + 1810836\rho^5 t_1^9 t_{11} - 656100t_1^{10} t_{11}$
$-314928\rho t_1^{10} t_{11} - 157464\rho^2 t_1^{10} t_{11} + 170586\rho^3 t_1^{10} t_{11} + 328050\rho^4 t_1^{10} t_{11} + 354294\rho^5 t_1^{10} t_{11}$
$- 17496t_1^{11} t_{11} + 4374\rho t_1^{11} t_{11} + 109350\rho^2 t_1^{11} t_{11} + 131220\rho^3 t_1^{11} t_{11} + 83106\rho^4 t_1^{11} t_{11} +$
$87480\rho^5 t_1^{11} t_{11} - 2916t_1^{12} t_{11} + 2916\rho t_1^{12} t_{11} + 13122\rho^2 t_1^{12} t_{11} + 8748\rho^3 t_1^{12} t_{11} +$
$4374\rho^4 t_1^{12} t_{11} - 5832\rho^5 t_1^{12} t_{11} + 486\rho t_1^{13} t_{11} + 486\rho^3 t_1^{13} t_{11} - 486\rho^4 t_1^{13} t_{11}$
$-486\rho^5 t_1^{13} t_{11} - 387420489\rho t_{11}^2 + 516560652\rho^3 t_1 t_{11}^2 + 516560652\rho^4 t_1 t_{11}^2 + 430467210t_1^2 t_{11}^2$
$+ 688747536\rho t_1^2 t_{11}^2 + 688747536\rho^2 t_1^2 t_{11}^2 + 688747536\rho^3 t_1^2 t_{11}^2 + 688747536\rho^4 t_1^2 t_{11}^2 +$
$430467210\rho^5 t_1^2 t_{11}^2 + 57395628t_1^3 t_{11}^2 + 344373768\rho t_1^3 t_{11}^2 + 344373768\rho^2 t_1^3 t_{11}^2$
$+57395628\rho^3 t_1^3 t_{11}^2 + 4782969t_1^4 t_{11}^2 + 4782969\rho t_1^4 t_{11}^2 - 71744535\rho^3 t_1^4 t_{11}^2 - 167403915\rho^4 t_1^4 t_{11}^2$
$-71744535\rho^5 t_1^4 t_{11}^2 - 31886460\rho t_1^5 t_{11}^2 - 38263752\rho^2 t_1^5 t_{11}^2 - 38263752\rho^3 t_1^5 t_{11}^2 - 38263752\rho^4 t_1^5 t_{11}^2$
$-31886460\rho^5 t_1^5 t_{11}^2 - 3188646\rho t_1^6 t_{11}^2 - 8503056\rho^2 t_1^6 t_{11}^2 - 3188646\rho^3 t_1^6 t_{11}^2 + 708588\rho^4 t_1^7 t_{11}^2 +$

$708588\rho^5 t_1^7 t_{11}^2 - 59049t_1^8 t_{11}^2 - 1549681956t_3 - 3099363912\rho t_3 - 2324522934\rho^2 t_3$
$- 1549681956\rho^4 t_3 - 2324522934\rho^5 t_3 + 258280326t_1 t_3 - 1807962282\rho t_1 t_3$
$- 774840978\rho^2 t_1 t_3 + 3615924564\rho^3 t_1 t_3 + 2582803260\rho^4 t_1 t_3 - 2066242608\rho^5 t_1 t_3 +$
$2496709818t_1^2 t_3 + 3529831122\rho t_1^2 t_3 + 3185457354\rho^2 t_1^2 t_3 + 5423886846\rho^3 t_1^2 t_3 +$
$4304672100\rho^4 t_1^2 t_3 - 258280326\rho^5 t_1^2 t_3 + 2955874842t_1^3 t_3 + 3271550796\rho t_1^3 t_3 +$
$3041968284\rho^2 t_1^3 t_3 + 2152336050\rho^3 t_1^3 t_3 + 1348797258\rho^4 t_1^3 t_3 - 1119214746\rho^5 t_1^3 t_3 +$
$1788830406t_1^4 t_3 + 1769698530\rho t_1^4 t_3 + 143489070\rho^2 t_1^4 t_3 - 411335334\rho^3 t_1^4 t_3 - 28697814\rho^4 t_1^4 t_3$
$-1320099444\rho^5 t_1^4 t_3 + 886443588t_1^5 t_3 + 924707340\rho t_1^5 t_3 - 197696052\rho^2 t_1^5 t_3 - 717445350\rho^3 t_1^5 t_3$
$-31886460\rho^4 t_1^5 t_3 + 229582512\rho^5 t_1^5 t_3 - 46766808t_1^6 t_3 - 100973790\rho t_1^6 t_3 - 177501294\rho^2 t_1^6 t_3$
$-640917846\rho^3 t_1^6 t_3 - 248714388\rho^4 t_1^6 t_3 + 61647156\rho^5 t_1^6 t_3 - 91053558t_1^7 t_3 - 157306536\rho t_1^7 t_3$
$-186004350\rho^2 t_1^7 t_3 - 161558064\rho^3 t_1^7 t_3 - 106288200\rho^4 t_1^7 t_3 - 21966228\rho^5 t_1^7 t_3 + 8621154t_1^8 t_3$
$-14526054\rho t_1^8 t_3 - 14053662\rho^2 t_1^8 t_3 + 1535274\rho^3 t_1^8 t_3 + 46058220\rho^4 t_1^8 t_3 + 66607272\rho^5 t_1^8 t_3$
$-18620118t_1^9 t_3 - 7518906\rho t_1^9 t_3 + 12282192\rho^2 t_1^9 t_3 + 3542940\rho^3 t_1^9 t_3 + 13581270\rho^4 t_1^9 t_3 +$
$19053144\rho^5 t_1^9 t_3 - 4566456t_1^{10} t_3 - 2768742\rho t_1^{10} t_3 + 5038848\rho^2 t_1^{10} t_3 + 9316620\rho^3 t_1^{10} t_3 +$
$1299078\rho^4 t_1^{10} t_3 + 1141614\rho^5 t_1^{10} t_3 + 1753974t_1^{11} t_3 + 1548396\rho t_1^{11} t_3 + 2934954\rho^2 t_1^{11} t_3 +$
$3831624\rho^3 t_1^{11} t_3 + 2370708\rho^4 t_1^{11} t_3 + 603612\rho^5 t_1^{11} t_3 + 237654t_1^{12} t_3 + 720252\rho t_1^{12} t_3 +$
$733374\rho^2 t_1^{12} t_3 + 542376\rho^3 t_1^{12} t_3 + 371790\rho^4 t_1^{12} t_3 + 37908\rho^5 t_1^{12} t_3 + 45198t_1^{13} t_3 +$
$93798\rho t_1^{13} t_3 + 104976\rho^2 t_1^{13} t_3 + 78732\rho^3 t_1^{13} t_3 - 52974\rho^4 t_1^{13} t_3 - 28188\rho^5 t_1^{13} t_3 +$
$18144t_1^{14} t_3 + 9072\rho t_1^{14} t_3 + 6156\rho^2 t_1^{14} t_3 + 2430\rho^3 t_1^{14} t_3 - 6642\rho^4 t_1^{14} t_3 - 13284\rho^5 t_1^{14} t_3 +$

$1296t_1^{15}t_3 + 1728\rho t_1^{15}t_3 - 1350\rho^2t_1^{15}t_3 - 1458\rho^3t_1^{15}t_3 - 432\rho^4t_1^{15}t_3 - 918\rho^5t_1^{15}t_3 +$
$54t_1^{16}t_3 + 54\rho t_1^{16}t_3 - 108\rho^2t_1^{16}t_3 - 108\rho^3t_1^{16}t_3 - 54\rho^4t_1^{16}t_3 + 162\rho^5t_1^{16}t_3 - 6\rho t_1^{17}t_3$
$- 6\rho^3t_1^{17}t_3 + 6\rho^4t_1^{17}t_3 + 6\rho^5t_1^{17}t_3 + 774840978\rho^3t_{11}t_3 + 774840978\rho^4t_{11}t_3$
$-774840978\rho^5t_{11}t_3+1291401630t_1t_{11}t_3+2066242608\rho t_1t_{11}t_3+1807962282\rho^2t_1t_{11}t_3+$

$2324522934\rho^3t_1t_{11}t_3+1549681956\rho^4t_1t_{11}t_3+516560652\rho^5t_1t_{11}t_3+1119214746t_1^2t_{11}t_3+$

$1980149166\rho t_1^2t_{11}t_3+2066242608\rho^2t_1^2t_{11}t_3+688747536\rho^3t_1^2t_{11}t_3+258280326\rho^4t_1^2t_{11}t_3$
$- 430467210\rho^5t_1^2t_{11}t_3 + 459165024t_1^3t_{11}t_3 + 803538792\rho t_1^3t_{11}t_3 + 57395628\rho^2t_1^3t_{11}t_3$
$-602654094\rho^3t_1^3t_{11}t_3-1262703816\rho^4t_1^3t_{11}t_3-947027862\rho^5t_1^3t_{11}t_3+325241892t_1^4t_{11}t_3$
$-153055008\rho t_1^4t_{11}t_3-535692528\rho^2t_1^4t_{11}t_3-822670668\rho^3t_1^4t_{11}t_3-822670668\rho^4t_1^4t_{11}t_3$
$- 420901272\rho^5t_1^4t_{11}t_3 - 70150212t_1^5t_{11}t_3 - 267846264\rho t_1^5t_{11}t_3 - 491051484\rho^2t_1^5t_{11}t_3$
$- 491051484\rho^3t_1^5t_{11}t_3 - 267846264\rho^4t_1^5t_{11}t_3 - 178564176\rho^5t_1^5t_{11}t_3 - 37200870t_1^6t_{11}t_3$
$-96722262\rho t_1^6t_{11}t_3-163683828\rho^2t_1^6t_{11}t_3-96722262\rho^3t_1^6t_{11}t_3-22320522\rho^4t_1^6t_{11}t_3+$
$37200870\rho^5t_1^6t_{11}t_3 - 14880348t_1^7t_{11}t_3 - 27280638\rho t_1^7t_{11}t_3 - 19840464\rho^2t_1^7t_{11}t_3$
$- 10274526\rho^3t_1^7t_{11}t_3 + 24446286\rho^4t_1^7t_{11}t_3 + 25154874\rho^5t_1^7t_{11}t_3 - 7203978t_1^8t_{11}t_3$
$- 1771470\rho t_1^8t_{11}t_3 + 826686\rho^2t_1^8t_{11}t_3 + 4723920\rho^3t_1^8t_{11}t_3 + 8384958\rho^4t_1^8t_{11}t_3 +$
$6377292\rho^5t_1^8t_{11}t_3-236196t_1^9t_{11}t_3+747954\rho t_1^9t_{11}t_3+2361960\rho^2t_1^9t_{11}t_3+2440692\rho^3t_1^9t_{11}t_3+$

$1535274\rho^4t_1^9t_{11}t_3+1574640\rho^5t_1^9t_{11}t_3-26244t_1^{10}t_{11}t_3+91854\rho t_1^{10}t_{11}t_3+341172\rho^2t_1^{10}t_{11}t_3$
$+183708\rho^3t_1^{10}t_{11}t_3+26244\rho^4t_1^{10}t_{11}t_3-118098\rho^5t_1^{10}t_{11}t_3+4374t_1^{11}t_{11}t_3+8748\rho t_1^{11}t_{11}t_3+$

$8748\rho^3t_1^{11}t_{11}t_3-21870\rho^4t_1^{11}t_{11}t_3-21870\rho^5t_1^{11}t_{11}t_3+1458t_1^{12}t_{11}t_3+774840978t_{11}^2t_3+$
$774840978\rho t_{11}^2t_3 + 774840978\rho^2t_{11}^2t_3 + 774840978\rho^3t_{11}^2t_3 + 774840978\rho^4t_{11}^2t_3 +$
$774840978\rho^5t_{11}^2t_3 + 774840978\rho t_1t_{11}^2t_3 + 774840978\rho^2t_1t_{11}^2t_3 - 258280326\rho^3t_1^2t_{11}^2t_3$
$-774840978\rho^4t_1^2t_{11}^2t_3-258280326\rho^5t_1^2t_{11}^2t_3-229582512\rho t_1^3t_{11}^2t_3-258280326\rho^2t_1^3t_{11}^2t_3$
$- 258280326\rho^3t_1^3t_{11}^2t_3 - 258280326\rho^4t_1^3t_{11}^2t_3 - 229582512\rho^5t_1^3t_{11}^2t_3 - 28697814\rho t_1^4t_{11}^2t_3$
$- 86093442\rho^2t_1^4t_{11}^2t_3 - 28697814\rho^3t_1^4t_{11}^2t_3 + 9565938\rho^4t_1^5t_{11}^2t_3 + 9565938\rho^5t_1^5t_{11}^2t_3$
$- 1062882t_1^6t_{11}^2t_3 - 1549681956t_3^2 - 774840978\rho t_3^2 - 387420489\rho^2t_3^2 + 387420489\rho^3t_3^2$
$-387420489\rho^4t_3^2-387420489\rho^5t_3^2-1678822119t_1t_3^2-1420541793\rho t_1t_3^2+903981141\rho^2t_1t_3^2+$

$1033121304\rho^3t_1t_3^2-645700815\rho^4t_1t_3^2-1549681956\rho^5t_1t_3^2+215233605t_1^2t_3^2-473513931\rho t_1^2t_3^2+$

$903981141\rho^2t_1^2t_3^2 + 1162261467\rho^3t_1^2t_3^2 - 516560652\rho^4t_1^2t_3^2 - 2238429492\rho^5t_1^2t_3^2 +$
$1018772397t_1^3t_3^2 + 459165024\rho t_1^3t_3^2 + 57395628\rho^2t_1^3t_3^2 + 143489070\rho^3t_1^3t_3^2$
$-258280326\rho^4t_1^3t_3^2-1219657095\rho^5t_1^3t_3^2+511777683t_1^4t_3^2-19131876\rho t_1^4t_3^2-243931419\rho^2t_1^4t_3^2$
$-521343621\rho^3t_1^4t_3^2-306110016\rho^4t_1^4t_3^2-325241892\rho^5t_1^4t_3^2+25509168t_1^5t_3^2-279006525\rho t_1^5t_3^2$
$-385826166\rho^2t_1^5t_3^2-347562414\rho^3t_1^5t_3^2-172186884\rho^4t_1^5t_3^2-191318760\rho^5t_1^5t_3^2+52612659t_1^6t_3^2$
$-58458510\rho t_1^6t_3^2-131265927\rho^2t_1^6t_3^2-23383404\rho^3t_1^6t_3^2+76527504\rho^4t_1^6t_3^2+72807417\rho^5t_1^6t_3^2$
$-1240029t_1^7t_3^2-2657205\rho t_1^7t_3^2+14703201\rho^2t_1^7t_3^2+17006112\rho^3t_1^7t_3^2+68378742\rho^4t_1^7t_3^2+$
$65367243\rho^5t_1^7t_3^2-10274526t_1^8t_3^2+1299078\rho t_1^8t_3^2+16947063\rho^2t_1^8t_3^2+22084326\rho^3t_1^8t_3^2+$
$15234642\rho^4t_1^8t_3^2 + 11101212\rho^5t_1^8t_3^2 + 4192479t_1^9t_3^2 + 5727753\rho t_1^9t_3^2 + 10058013\rho^2t_1^9t_3^2 +$
$11868849\rho^3t_1^9t_3^2 + 5747436\rho^4t_1^9t_3^2 + 2401326\rho^5t_1^9t_3^2 + 1371249t_1^{10}t_3^2 + 2453814\rho t_1^{10}t_3^2 +$
$2775303\rho^2t_1^{10}t_3^2 + 1863324\rho^3t_1^{10}t_3^2 + 807003\rho^4t_1^{10}t_3^2 - 360855\rho^5t_1^{10}t_3^2 + 301806t_1^{11}t_3^2 +$
$422091\rho t_1^{11}t_3^2 + 279936\rho^2t_1^{11}t_3^2 + 146529\rho^3t_1^{11}t_3^2 - 334611\rho^4t_1^{11}t_3^2 - 295245\rho^5t_1^{11}t_3^2 +$
$105705t_1^{12}t_3^2+35721\rho t_1^{12}t_3^2-22599\rho^2t_1^{12}t_3^2-45198\rho^3t_1^{12}t_3^2-64152\rho^4t_1^{12}t_3^2-70713\rho^5t_1^{12}t_3^2+$

$6561t_1^{13}t_3^2 + 2916\rho t_1^{13}t_3^2 - 19926\rho^2t_1^{13}t_3^2 - 19926\rho^3t_1^{13}t_3^2 - 6075\rho^4t_1^{13}t_3^2 - 6075\rho^5t_1^{13}t_3^2 +$

$324t_1^{14}t_3^2 - 324\rho t_1^{14}t_3^2 - 2106\rho^2 t_1^{14}t_3^2 - 1620\rho^3 t_1^{14}t_3^2 + 1944\rho^5 t_1^{14}t_3^2 - 54t_1^{15}t_3^2$
$- 108\rho t_1^{15}t_3^2 - 108\rho^3 t_1^{15}t_3^2 + 162\rho^4 t_1^{15}t_3^2 + 162\rho^5 t_1^{15}t_3^2 - 9t_1^{16}t_3^2 + 774840978\rho^2 t_{11}t_3^2 +$
$774840978\rho^4 t_{11}t_3^2 - 258280326t_1t_{11}t_3^2 + 516560652\rho t_1t_{11}t_3^2 + 516560652\rho^2 t_1t_{11}t_3^2 +$
$258280326\rho^3 t_1t_{11}t_3^2 - 258280326\rho^4 t_1t_{11}t_3^2 - 516560652\rho^5 t_1t_{11}t_3^2 + 516560652t_1^2t_{11}t_3^2$
$-258280326\rho^3 t_1^2t_{11}t_3^2 - 602654094\rho^4 t_1^2t_{11}t_3^2 - 344373768\rho^5 t_1^2t_{11}t_3^2 - 315675954\rho t_1^3t_{11}t_3^2$
$-516560652\rho^2 t_1^3t_{11}t_3^2 - 545258466\rho^3 t_1^3t_{11}t_3^2 - 200884698\rho^4 t_1^3t_{11}t_3^2 - 286978140\rho^5 t_1^3t_{11}t_3^2$
$- 38263752t_1^4t_{11}t_3^2 - 86093442\rho t_1^4t_{11}t_3^2 - 210450636\rho^2 t_1^4t_{11}t_3^2 - 76527504\rho^3 t_1^4t_{11}t_3^2 +$
$47829690\rho^4 t_1^4t_{11}t_3^2 + 95659380\rho^5 t_1^4t_{11}t_3^2 - 9565938t_1^5t_{11}t_3^2 - 22320522\rho t_1^5t_{11}t_3^2 +$
$6377292\rho^2 t_1^5t_{11}t_3^2 + 9565938\rho^3 t_1^5t_{11}t_3^2 + 76527504\rho^4 t_1^5t_{11}t_3^2 + 73338858\rho^5 t_1^5t_{11}t_3^2$
$- 17006112t_1^6t_{11}t_3^2 + 2125764\rho t_1^6t_{11}t_3^2 + 12754584\rho^2 t_1^6t_{11}t_3^2 + 14880348\rho^3 t_1^6t_{11}t_3^2 +$
$23383404\rho^4 t_1^6t_{11}t_3^2 + 15943230\rho^5 t_1^6t_{11}t_3^2 - 354294t_1^7t_{11}t_3^2 + 3897234\rho t_1^7t_{11}t_3^2 +$
$9920232\rho^2 t_1^7t_{11}t_3^2 + 8857350\rho^3 t_1^7t_{11}t_3^2 + 4251528\rho^4 t_1^7t_{11}t_3^2 + 4960116\rho^5 t_1^7t_{11}t_3^2 +$
$118098t_1^8t_{11}t_3^2 + 472392\rho t_1^8t_{11}t_3^2 + 1889568\rho^2 t_1^8t_{11}t_3^2 + 826686\rho^3 t_1^8t_{11}t_3^2$
$- 236196\rho^4 t_1^8t_{11}t_3^2 - 708588\rho^5 t_1^8t_{11}t_3^2 + 78732t_1^9t_{11}t_3^2 + 39366\rho t_1^9t_{11}t_3^2 + 39366\rho^3 t_1^9t_{11}t_3^2$
$- 196830\rho^4 t_1^9t_{11}t_3^2 - 196830\rho^5 t_1^9t_{11}t_3^2 + 26244t_1^{10}t_{11}t_3^2 - 387420489\rho^4 t_{11}^2t_3^2$
$-258280326\rho t_1t_{11}^2t_3^2 - 258280326\rho^2 t_1t_{11}^2t_3^2 - 258280326\rho^3 t_1t_{11}^2t_3^2 - 258280326\rho^4 t_1t_{11}^2t_3^2$
$- 258280326\rho^5 t_1t_{11}^2t_3^2 - 43046721\rho t_1^2t_{11}^2t_3^2 - 172186884\rho^2 t_1^2t_{11}^2t_3^2 - 43046721\rho^3 t_1^2t_{11}^2t_3^2$
$+ 28697814\rho^4 t_1^3t_{11}^2t_3^2 + 28697814\rho^5 t_1^3t_{11}^2t_3^2 - 4782969t_1^4t_{11}^2t_3^2 - 774840978t_3^3$
$-258280326t_1t_3^3 + 258280326\rho t_1t_3^3 + 774840978\rho^2 t_1t_3^3 + 774840978\rho^3 t_1t_3^3 + 258280326\rho^5 t_1t_3^3$
$+ 172186884\rho t_1^2t_3^3 + 602654094\rho^2 t_1^2t_3^3 + 344373768\rho^3 t_1^2t_3^3 - 172186884\rho^4 t_1^2t_3^3$
$- 344373768\rho^5 t_1^2t_3^3 + 57395628t_1^3t_3^3 + 114791256\rho t_1^3t_3^3 + 114791256\rho^2 t_1^3t_3^3 +$
$86093442\rho^3 t_1^3t_3^3 - 258280326\rho^4 t_1^3t_3^3 - 315675954\rho^5 t_1^3t_3^3 + 86093442t_1^4t_3^3 +$
$38263752\rho t_1^4t_3^3 + 9565938\rho^2 t_1^4t_3^3 - 9565938\rho^3 t_1^4t_3^3 - 124357194\rho^4 t_1^4t_3^3 - 133923132\rho^5 t_1^4t_3^3 +$

$28697814t_1^5t_3^3 + 6377292\rho t_1^5t_3^3 - 25509168\rho^2 t_1^5t_3^3 - 31886460\rho^3 t_1^5t_3^3 - 51018336\rho^4 t_1^5t_3^3$
$- 60584274\rho^5 t_1^5t_3^3 + 9565938t_1^6t_3^3 + 1062882\rho t_1^6t_3^3 - 14880348\rho^2 t_1^6t_3^3 - 13817466\rho^3 t_1^6t_3^3$
$- 17006112\rho^4 t_1^6t_3^3 - 17006112\rho^5 t_1^6t_3^3 + 3188646t_1^7t_3^3 - 4960116\rho^2 t_1^7t_3^3 - 4960116\rho^3 t_1^7t_3^3$
$- 4960116\rho^4 t_1^7t_3^3 - 4960116\rho^5 t_1^7t_3^3 + 944784t_1^8t_3^3 - 118098\rho t_1^8t_3^3 - 2007666\rho^2 t_1^8t_3^3$
$- 2007666\rho^3 t_1^8t_3^3 - 1653372\rho^4 t_1^8t_3^3 - 1771470\rho^5 t_1^8t_3^3 + 275562t_1^9t_3^3 - 78732\rho t_1^9t_3^3$
$- 826686\rho^2 t_1^9t_3^3 - 708588\rho^3 t_1^9t_3^3 - 472392\rho^4 t_1^9t_3^3 - 393660\rho^5 t_1^9t_3^3 + 65610t_1^{10}t_3^3$
$- 52488\rho t_1^{10}t_3^3 - 236196\rho^2 t_1^{10}t_3^3 - 223074\rho^3 t_1^{10}t_3^3 - 65610\rho^4 t_1^{10}t_3^3 - 39366\rho^5 t_1^{10}t_3^3$
$- 8748t_1^{11}t_3^3 - 17496\rho t_1^{11}t_3^3 - 65610\rho^2 t_1^{11}t_3^3 - 56862\rho^3 t_1^{11}t_3^3 - 4374\rho^4 t_1^{11}t_3^3 - 2916t_1^{12}t_3^3$
$- 2916\rho t_1^{12}t_3^3 - 8748\rho^2 t_1^{12}t_3^3 - 5832\rho^3 t_1^{12}t_3^3 + 2916\rho^4 t_1^{12}t_3^3 + 7290\rho^5 t_1^{12}t_3^3$
$- 972t_1^{13}t_3^3 - 486\rho t_1^{13}t_3^3 - 486\rho^3 t_1^{13}t_3^3 + 972\rho^4 t_1^{13}t_3^3 + 972\rho^5 t_1^{13}t_3^3 - 162t_1^{14}t_3^3$
$+774840978\rho^2 t_{11}t_3^3 + 258280326\rho^2 t_1t_{11}t_3^3 - 258280326\rho^4 t_1t_{11}t_3^3 - 258280326\rho^5 t_1t_{11}t_3^3 +$

$86093442t_1^2t_{11}t_3^3 + 86093442\rho^2 t_1^2t_{11}t_3^3 - 86093442\rho^4 t_1^2t_{11}t_3^3 - 86093442\rho^5 t_1^2t_{11}t_3^3 +$
$28697814t_1^3t_{11}t_3^3 + 28697814\rho^2 t_1^3t_{11}t_3^3 - 28697814\rho^4 t_1^3t_{11}t_3^3 - 28697814\rho^5 t_1^3t_{11}t_3^3 +$
$9565938t_1^4t_{11}t_3^3 + 9565938\rho^2 t_1^4t_{11}t_3^3 - 9565938\rho^4 t_1^4t_{11}t_3^3 - 9565938\rho^5 t_1^4t_{11}t_3^3 +$
$3188646t_1^5t_{11}t_3^3 + 3188646\rho^2 t_1^5t_{11}t_3^3 - 3188646\rho^4 t_1^5t_{11}t_3^3 - 3188646\rho^5 t_1^5t_{11}t_3^3 +$
$1062882t_1^6t_{11}t_3^3 + 1062882\rho^2 t_1^6t_{11}t_3^3 - 1062882\rho^4 t_1^6t_{11}t_3^3 - 1062882\rho^5 t_1^6t_{11}t_3^3 +$
$354294t_1^7t_{11}t_3^3 - 354294\rho^4 t_1^7t_{11}t_3^3 - 354294\rho^5 t_1^7t_{11}t_3^3 + 118098t_1^8t_{11}t_3^3 - 387420489t_3^4$
$- 258280326t_1t_3^4 - 129140163t_1^2t_3^4 - 57395628t_1^3t_3^4 - 23914845t_1^4t_3^4 - 9565938t_1^5t_3^4$
$- 3720087t_1^6t_3^4 - 1062882t_1^7t_3^4 - 295245t_1^8t_3^4 - 78732t_1^9t_3^4 - 19683t_1^{10}t_3^4$
$- 4374t_1^{11}t_3^4 - 729t_1^{12}t_3^4 = 0.$

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